Notes on this practice problem set and the final exam

March 14, 2020

- The practice problems below are structured in a similar format to how the problems on the final exam might look like. It is strongly recommended to make a serious attempt at solving the practice problems yourself before looking at the solutions.[1]
- For problems requiring proofs, a correct solution is a proof written in words. Try to avoid symbols like $\implies$, $\forall$, $\therefore$ etc. and write instead using complete, grammatically correct (if possible) sentences.
- The material to be covered on the final exam consists of all the material discussed in the class and discussion section and contained in the summary notes, except for the topic of inner product spaces (Chapter 9 in the textbook, and the beginning of Chapter 11). Do not try to guess which specific topics will be asked about based on the topics of the problems in this problem set.
- The final exam will consist of 3–5 problems of a length and difficulty level roughly represented by the problems in this problem set.

[1] Solutions will be posted online tomorrow, Sunday 3/15.

Problem 1

You are given a system of 4 linear equations in 5 unknowns:

$$\begin{cases} 2x_1 - 4x_2 + 2x_3 \phantom{{}-x_4} + 6x_5 = 6 \\ \phantom{2x_1 - 4x_2 +{}} x_3 - x_4 \phantom{{}+6x_5} = -1 \\ 2x_1 - 4x_2 - x_3 + x_4 + 8x_5 = 7 \\ \phantom{2x_1 - 4x_2 +{}} x_3 + x_4 - 2x_5 = 1 \end{cases}$$

(a) Represent the system as an augmented matrix.

(b) Use the Gaussian elimination method to bring the augmented matrix to Reduced Row-Echelon Form (RREF).

(c) Use the RREF obtained in part (b) above to write the general form of the solution to the original system.

(d) Use the general form of the solution obtained in part (c) to write a specific solution to the system. That is, write specific numbers x_1, x_2, x_3, x_4, x_5 that solve the system. Substitute the numbers into the system to verify that they actually satisfy the equations.

Problem 2

(a) Find a basis for $\mathbb{R}^2$ of eigenvectors of the matrix $\begin{pmatrix} 8 & -3 \\ -3 & 0 \end{pmatrix}$.

(b) Given real numbers a, b, c, find a formula for the eigenvalues λ_1, λ_2 of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Show that the eigenvalues are always real numbers, and characterize for what values of the parameters a, b, c is it true that $\lambda_1 = \lambda_2$.

(c) Define what it means for a matrix to be diagonalizable.

(d) Prove that the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ (where a, b, c are real numbers as above) is always diagonalizable.

Hint: divide into two cases according to whether $\lambda_1 = \lambda_2$ or $\lambda_1 \neq \lambda_2$.

Problem 3

Let A be the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix}.$$

(a) Compute A^{-1}.

(b) Multiply the matrix you obtained in part (a) above by A to check that it is indeed the inverse matrix of A.

(c) Find all the eigenvalues of A.

Hint: you should be able to answer this without any computations (or with a very short computation that will make the answer obvious).

(d) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

Problem 4

(a) Compute the following determinants:

i. $\det\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 5 & 0 & 0 & 0 \end{pmatrix}$

ii. $\det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

iii. $\det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$ with $\overbrace{\hphantom{1 \; 1 \; 1 \; \cdots \; 1}}^{n \text{ columns}}$ (note: the answer may depend on n)

(b) Let A, B be square matrices of order n. Assume that B is invertible. Prove that

$$\det(BAB^{-1}) = \det(A).$$

(c) A square matrix A of order n is called *anti-symmetric* if it satisfies the condition

$$A^\top = -A$$

For example, the matrix

$$\begin{pmatrix} 0 & 5 & -1 \\ -5 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}$$

is anti-symmetric. Prove that if n is an odd number and A is an anti-symmetric matrix of order n then A is not invertible. (**Hint:** use determinants.)

Problem 5

(a) If a linear operator $T : V \to V$ has two eigenvectors v_1, v_2. Assume that the associated eigenvalues λ_1 and λ_2 are distinct. Are v_1, v_2 necessarily linearly independent? Prove that they are, or give an example that shows they don't have to be.

(b) Let V be a vector space with $\dim(V) = 3$, and let $T : V \to V$ be a linear operator on V. Assume that v_1, v_2, v_3 are eigenvectors of T, with associated eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 1.$$

Assume also that v_1 and v_2 are linearly independent. Prove that $\{v_1, v_2, v_3\}$ is a basis of V.

Hint: If v_1, v_2, v_3 are linearly dependent, show that you can find linearly dependent eigenvectors u, w such that $T(u) = \lambda_1 u$ and $T(w) = \lambda_3 w$, and explain why this leads to a contradiction.