Math 25 — Solutions to Homework Assignment #7

1. Prove that the sequence

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

converges.

Proof. We will apply the monotone convergence theorem. Note that since $\frac{2n-1}{2n} < 1$ we have that $a_{n+1} < a_n$. So the sequence $\{a_n\}$ is decreasing monotonically. Clearly, the sequence is bounded below by zero. So, by the monotone convergence theorem, it must converge. □

2. Define a sequence $\{x_n\}$ by

$$x_1 = \sqrt{3}, \quad x_2 = \sqrt{3+\sqrt{3}}, \quad x_{n+1} = \sqrt{3+x_n}.$$

Prove that the sequence converges and find its limit. For a small bonus credit, answer the same question when 3 is replaced an arbitrary integer $k \geq 2$.

Proof. We show that the sequence converges by applying the monotone convergence theorem. We will proceed by induction. First, we claim that $x_n < 3$ for all n. Note that $\sqrt{3} < 3$. Now, suppose that $x_n < 3$. Then,

$$x_{n+1} = \sqrt{3+x_n} \leq \sqrt{3+3} = \sqrt{6} < 3,$$

which shows the the sequence x_n is bounded above.

Next, note that $\sqrt{3} < \sqrt{3+\sqrt{3}}$. Now, suppose $x_n \geq x_{n-1}$. Then,

$$x_{n+1} = \sqrt{3+x_n} \leq \sqrt{3+x_{n-1}} = x_n,$$

which shows that the sequence is increasing. So we know that $x_n \to L$. Now, note that since $x_n = \sqrt{3+x_{n-1}}$, we also have $x_n^2 = 3 + x_{n-1}$. Then,

$$\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty}(3+x_{n-1}) \iff \lim_{n\to\infty} x_n^2 = 3 + \lim_{n\to\infty} x_{n-1} \iff L^2 = 3 + L \iff L^2 - L - 3 = 0.$$

Applying the quadratic formula, we see that

$$L = \frac{1 \pm \sqrt{1+4\cdot 3}}{2} = \frac{1 \pm \sqrt{13}}{2}.$$

Disregarding the negative solution, we obtain $L = \frac{1+\sqrt{13}}{2}$.

To show this result for arbitrary k, one may simply replace 3 by k in the above proof to obtain $L = \frac{1+\sqrt{1+4k}}{2}$. □

3. Define the sequence $\{a_n\}$ by $a_n = \left(1 + \frac{1}{n}\right)^n$.

(a) Show that $\{a_n\}$ is increasing and bounded from above. Define the constant e by $e := \lim_{n\to\infty} a_n$.

Proof. Applying the binomial theorem, we obtain

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1-\frac{1}{n}}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\cdots \leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots < 1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots \leq 3,$$

from which we can see that the sequence $\{a_n\}$ is bounded above, and also that it is increasing (the first expression for a_n shows that it is a sum of $n+1$ terms. Each of them is smaller than the corresponding terms in the expansion of a_{n+1}, except the first two terms $1+1$, which are equal; in addition, the expansion of a_{n+1} contains one more term, which is positive). So it must converge. □

(b) Show that $2 < e < 3$.

Proof. Since the sequence is strictly increasing and $a_1 = 2$, the lower bound is clear. That 3 is an upper bound was shown above. □

4. In an analog clock, at twelve o'clock both the hours dial and the minutes dial are pointing in the same direction. In this problem, we will find the next time that the two dials are aligned in two different ways.

(a) Let X be the number of minutes until the two dials point in the same direction again. Write an algebraic equation that X satisfies, and solve it to find X.

Proof. Let $\theta_m(t)$ be the angle of the minute hand at t minutes and $theta_h(t)$ be the angle of the hour hand at t minutes, where angles are measured counterclockwise from the positions of the hands at noon. Then we see that

$$\theta_m(t) = \frac{2\pi}{60}t \qquad \text{and} \qquad \theta_h(t) = \frac{2\pi}{720}t.$$

Now, if we just equate these two expressions, we will obtain the solution $t = 0$, which corresponds to a time of 12 o'clock. Since $\theta + 2\pi = \theta$, though, we can instead solve

$$\theta_m(t) = \theta_h(t) + 2\pi.$$

Note that $k \cdot 2\pi$ will yield the kth time after noon that the hands will cross. Now, solving the equation above, we obtain

$$\frac{2\pi}{60}t = \frac{2\pi}{720}t + 2\pi \iff 12t = t + 720 \iff 11t = 720 \iff t = \frac{720}{11}.$$

This is equal to one hour, five minutes and 5/11-ths of a minute (approx. 27 seconds). □

(b) Another way to compute X is by representing it as the sum of an infinite geometric series. Find the sum of the series you obtain.

Proof. Again, it is useful to think of the problem in terms of angles. In this way, we see that

$$\begin{aligned}
T_0 &= 0 \\
T_1 &= \frac{\pi}{6} \\
T_2 &= \frac{\pi}{6} + \frac{\pi}{72} \\
T_3 &= \frac{\pi}{6} + \frac{\pi}{72} + \frac{\pi}{864} \\
&\vdots \\
T_n &= 2\pi \sum_{k=1}^{n} \frac{1}{12^k}.
\end{aligned}$$

Then, applying what we know about geometric series, we see that $T_n \to \frac{24\pi}{11}$. Finally, using our expression for θ_m, we see that this corresponds to $\frac{720}{11}$ minutes, as before. $\square$