Quantum mushrooms, scars, and the high-frequency limit of chaotic eigenfunctions

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Planar Dirichlet eigenproblem

Normal modes of elastic membrane or ‘drum’ (Helmholtz, Germain, 19^{th}C)

Eigenfunctions ϕ_j of Laplacian $\Delta := \partial^2_{x_1} + \partial^2_{x_2}$ in bounded cavity $\Omega \subset \mathbb{R}^2$

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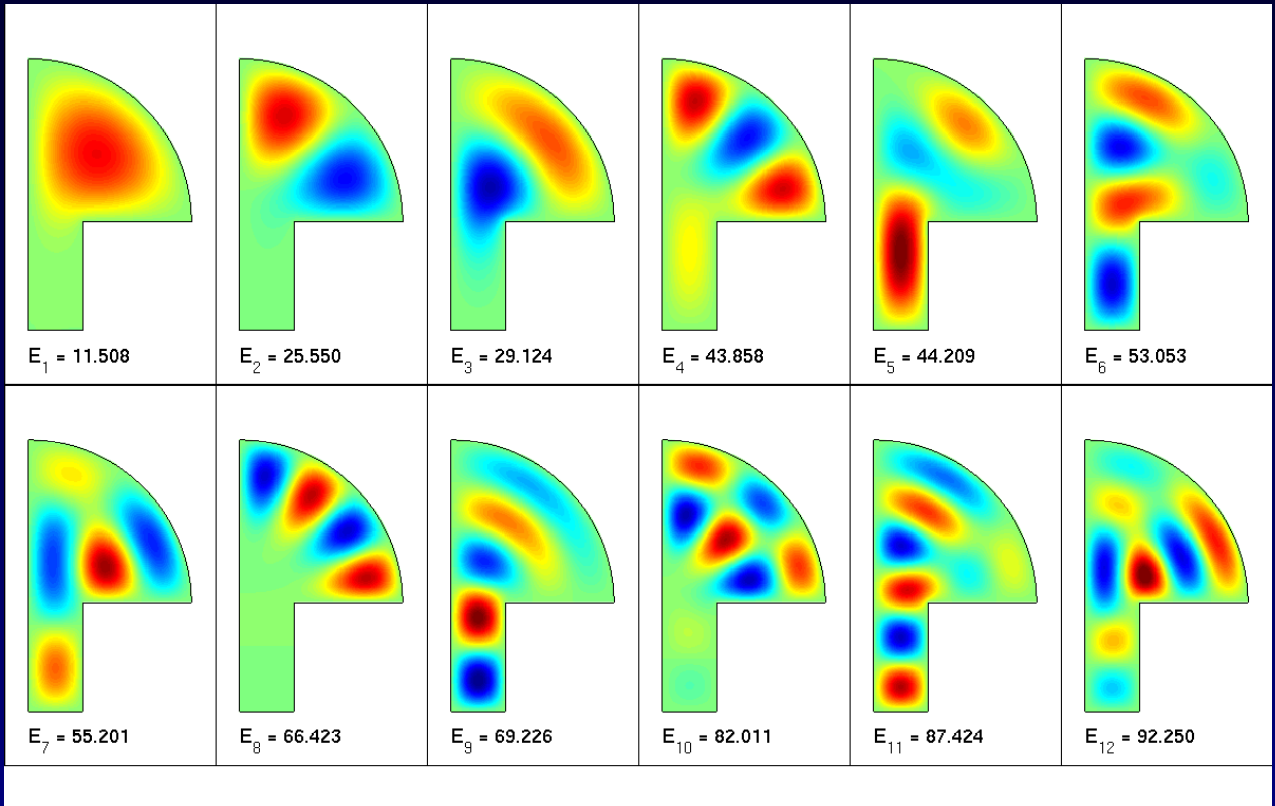

mode $j = 1 \cdots \infty$

discrete eigenvalues $E_1 < E_2 \leq E_3 \leq \cdots \infty$

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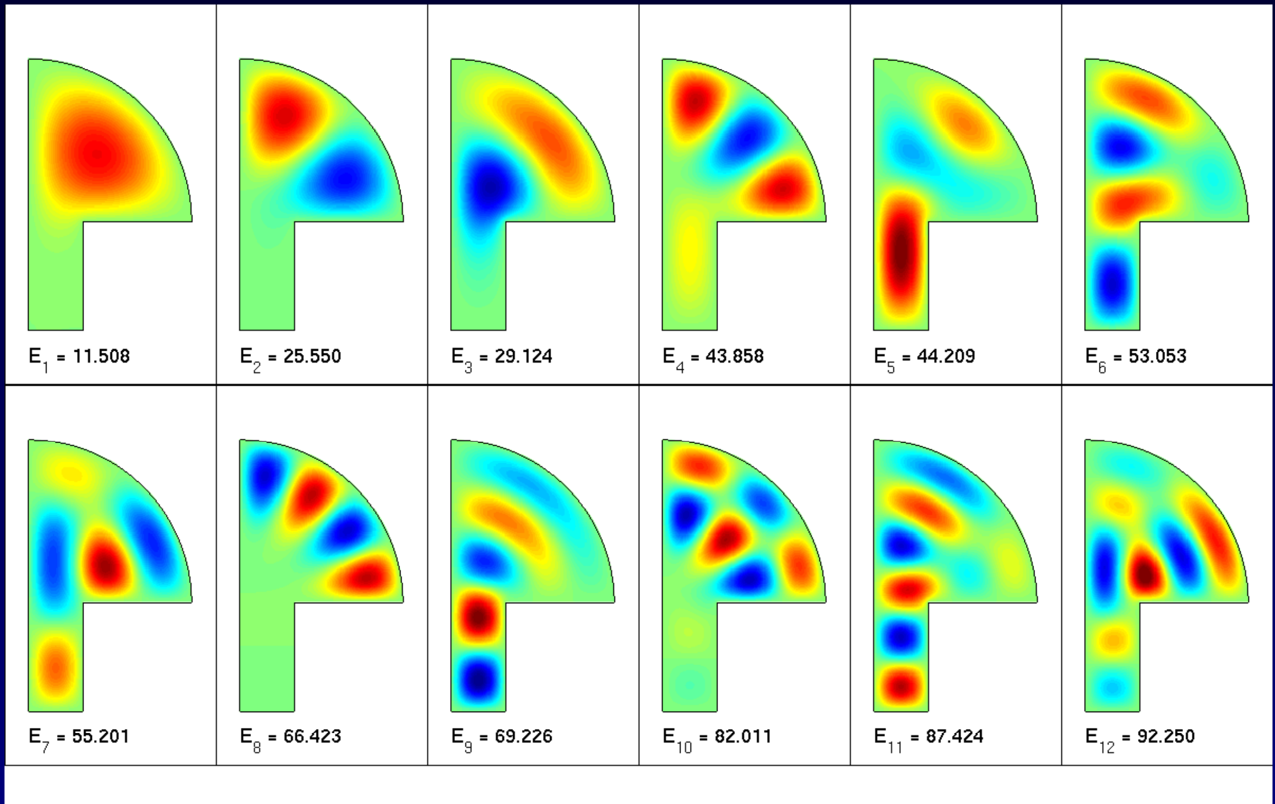

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- Time-harmonic solns of wave eqn (acoustics, optics, quantum, etc)
- Asymptotics of ϕ_j as eigenvalue $E_j \to \infty$? Depends on shape…

(Some favorite) eigenmode topics

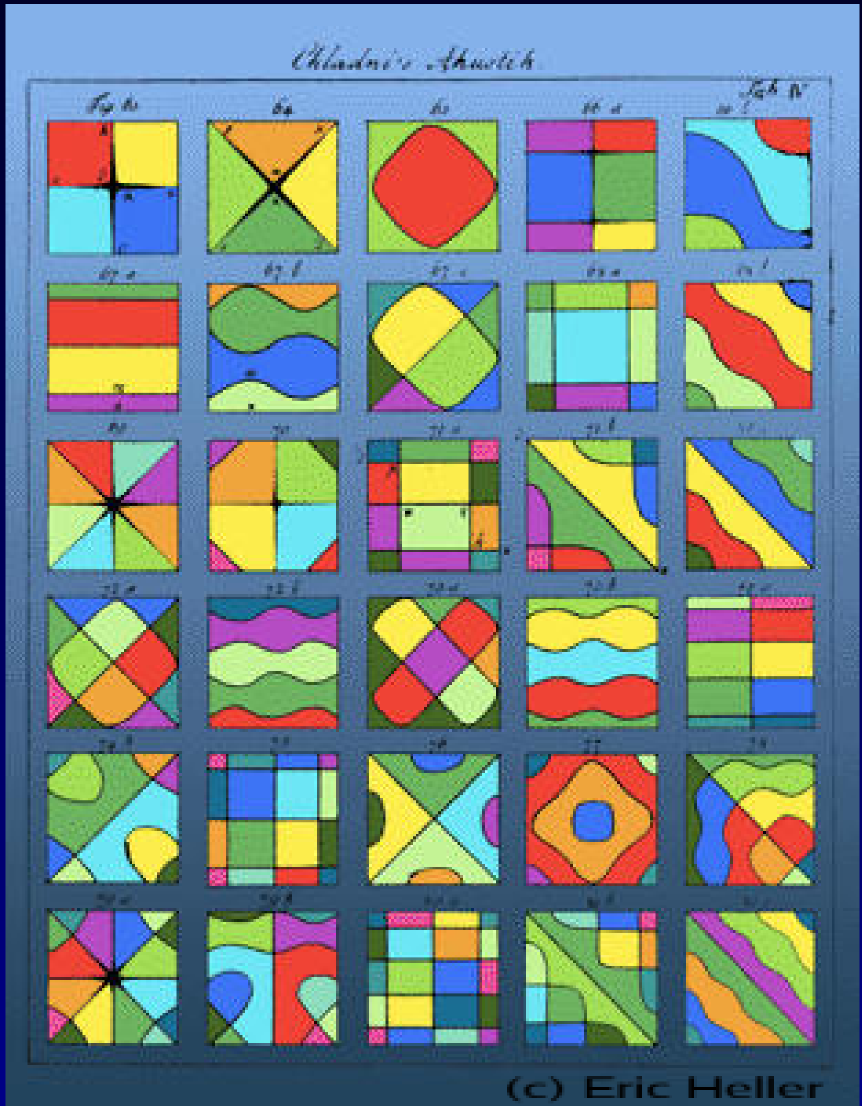
I. Background, chaotic billiards

II. Quantum ergodicity:
how uniform are eigenmodes?

III. Scarring and the mushroom:
how do periodic ray orbits affect mode statistics?

Some history of (related) eigenmodes

Ernst Chladni (1756–1827) sprinkles sand on metal plates
'Plays' them with violin bow: visualizes nodal lines ($\phi_j = 0$)



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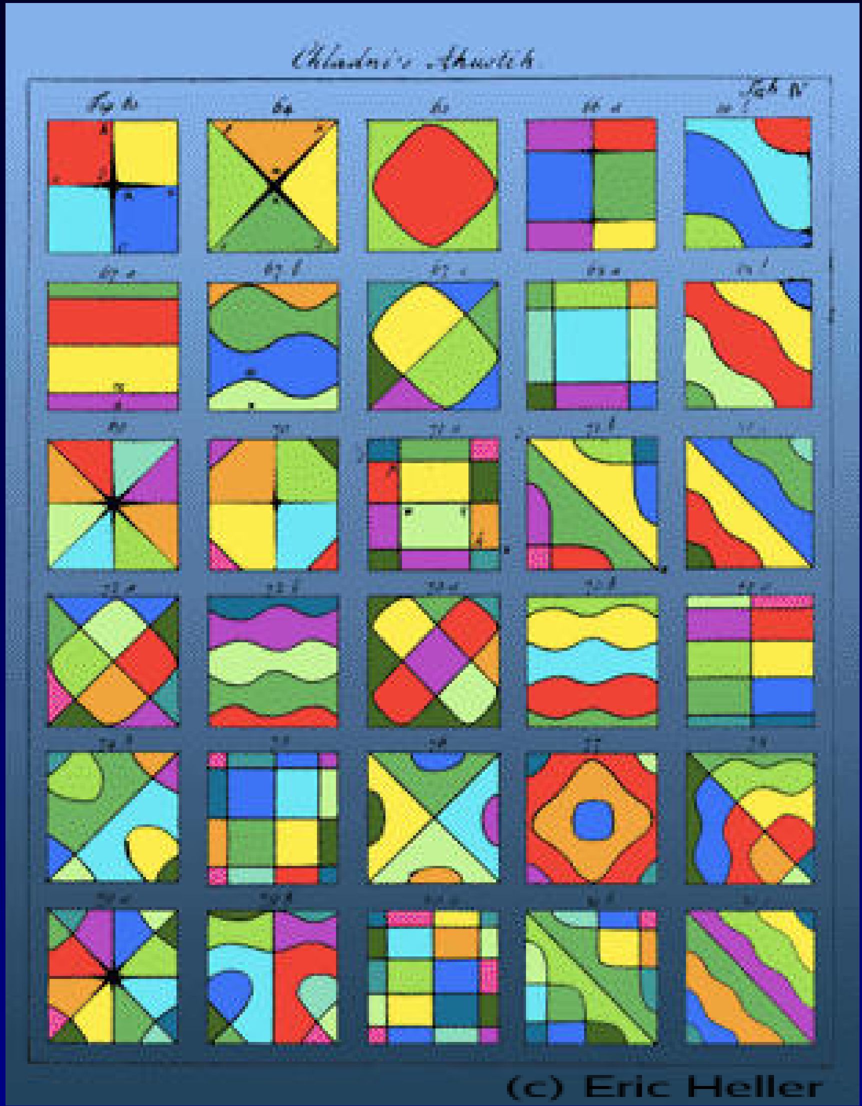

- popular lectures all over Europe
- Napoleon impressed: offers 1 kg gold to explain patterns
- Napoleon realised: irregularly shaped plate harder to understand

  (first funding for quantum chaos!)
- Sophie Germain got prize in 1816

Note: rigid plate $\neq$ membrane
(biharmonic Δ^2 rather than Laplacian Δ)

Einstein-Brillouin-Keller (EBK) (Keller '60)

Recall in separable domains (ellipse, rect.): ϕ_j products of 1D modes

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mode ≈ sum of traveling waves, 'rays'

S_m = phase function

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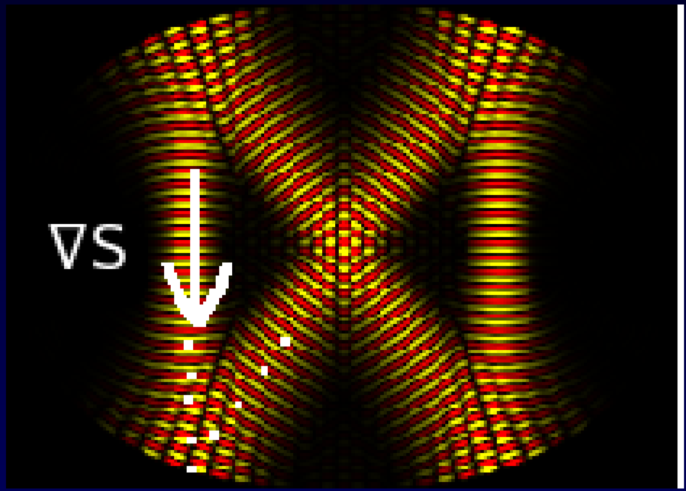

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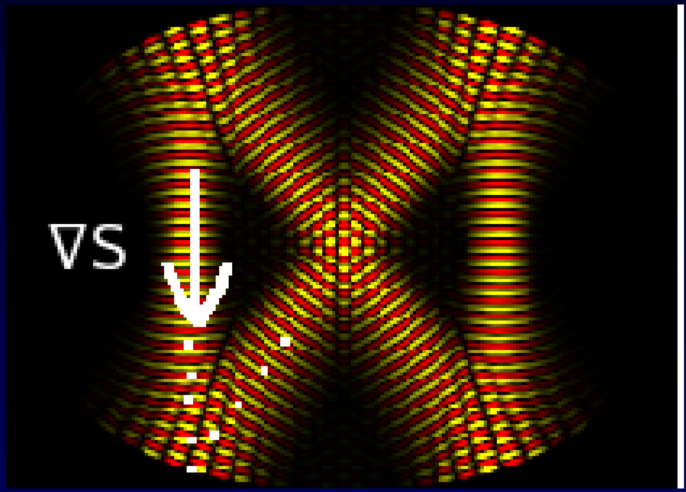

Insert into $-\Delta\phi = E\phi$ gives $|\nabla S_m| = 1$ phase grows along straight rays

For single-valued ϕ to exist and satisfy BCs:

i) rays reflect off boundary, giving ray families which must close

ii) quantization: round-trip phase $= 2\pi n + \frac{\pi}{2}$(# focal points) $+ \pi$(# reflections)

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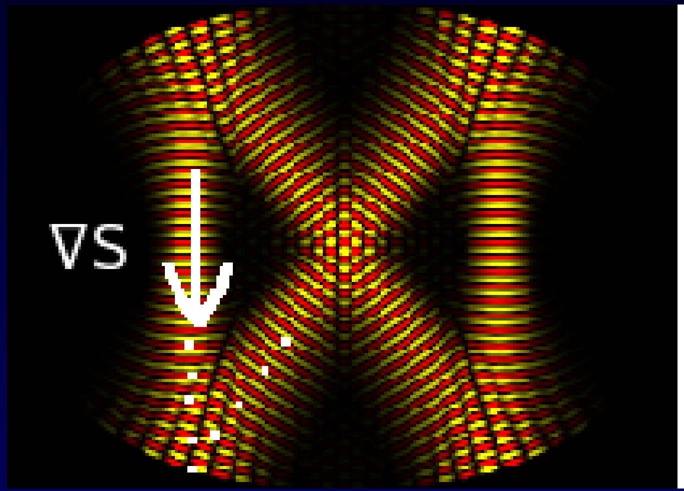

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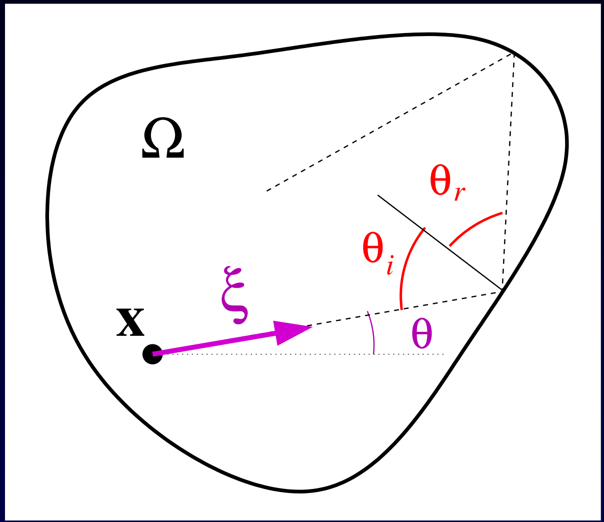
But, do bouncing ray paths always form closed families. . . ?

Bouncing rays: the game of billiards



full phase space $(x_1, x_2, \xi_1, \xi_2) =: (\mathbf{x}, \xi)$

free motion: Hamiltonian dynamical system
energy $H(\mathbf{x}, \xi) = |\xi|^2$ conserved

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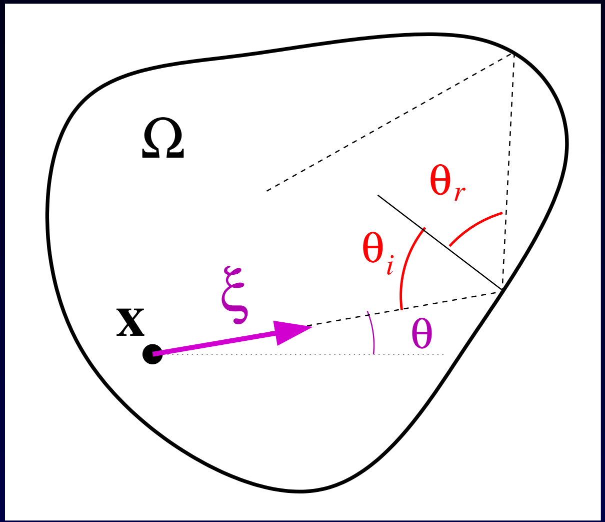

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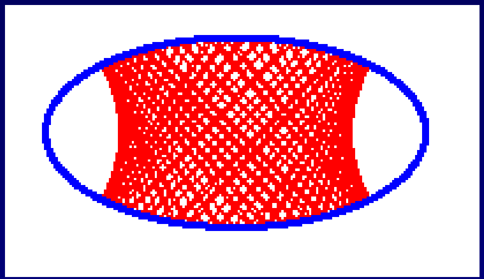
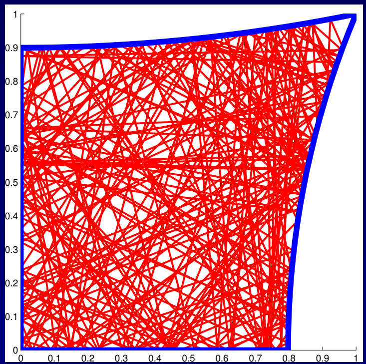
TWO BROAD CLASSES OF MOTION

integrable: 

d conserved quantities (d=2)

ergodic: 

only H conserved: chaos!

Properties of chaotic rays

Defn. of ergodic: Given $A(\mathbf{x})$ test func, for a.e. trajectory $\mathbf{x}_0$,

time average = spatial average

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T A(\mathbf{x}(t))dt \quad = \quad \frac{1}{\mathrm{vol}(\Omega)}\int_\Omega A(\mathbf{x})d\mathbf{x} =: \overline{A}$$

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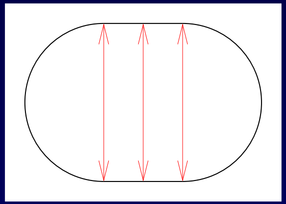
- 'hyperbolic': $|\mathbf{x}_1(t) - \mathbf{x}_2(t)| \sim ce^{\Lambda t}$ $\quad 0 < \Lambda =$ Lyapunov exponent
- 'Anosov': uniformly hyperbolic (all periodic orbits unstable, isolated)

e.g. Bunimovich stadium ergodic but not Anosov. . .



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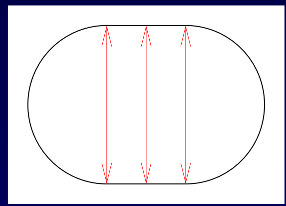

'QUANTUM CHAOS': study of eigenmodes when Ω ergodic (Einstein 1917)

- Apps: quantum dots, nano-scale devices, molecular phys/chem

Why chaos important? Generic shapes have some chaotic phase space

Modes ϕ_j irregular: VIEW $j \approx 3000$: 45 wavelengths across …say more?

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study mode matrix elements $\langle \phi_j, A\phi_j \rangle := \int_\Omega A(\mathbf{x})|\phi_j(\mathbf{x})|^2 d\mathbf{x}$

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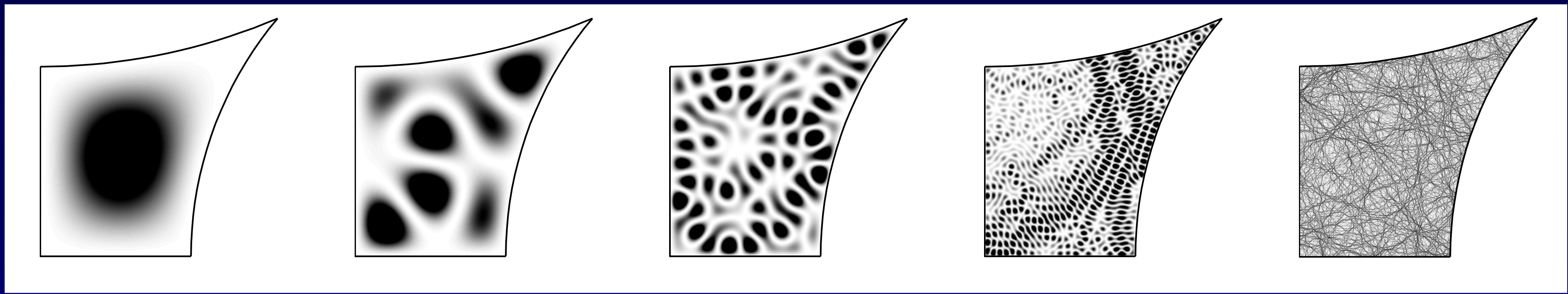

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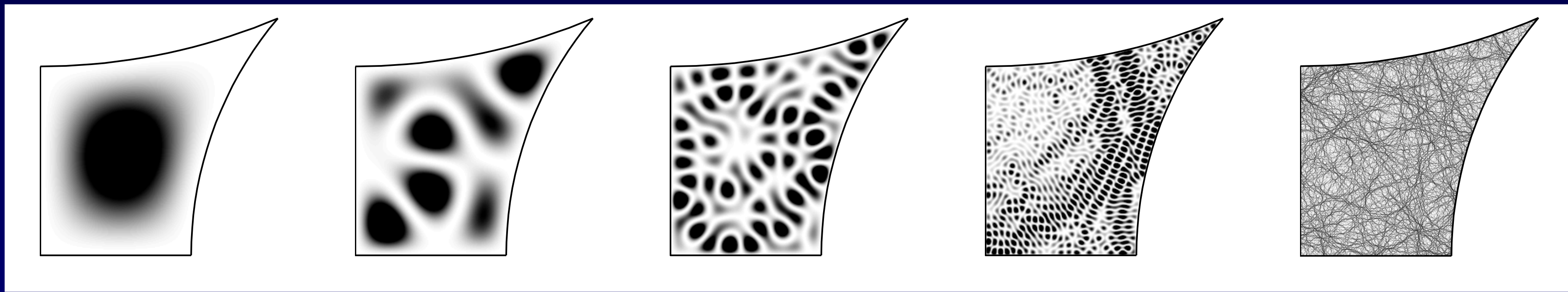

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Of practical importance: At what *rate* is limit reached?
How fast does the density of excluded subsequence vanish?

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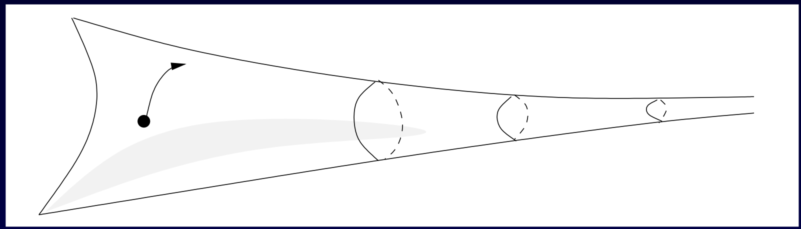

Recent analytic results:

- Proven to hold for *arithmetic* manifold $SL_2(\mathbb{Z}) \setminus \mathbb{H}$ (Lindenstrauss '03)

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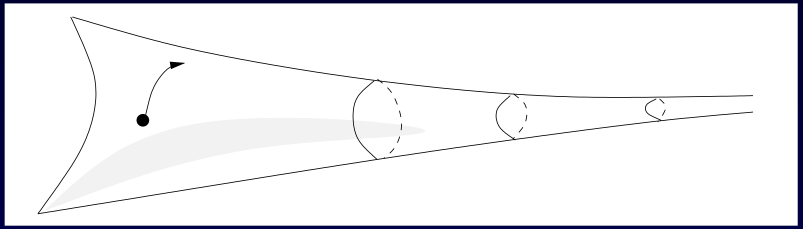

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But there are no analytic results for planar cavities. . .

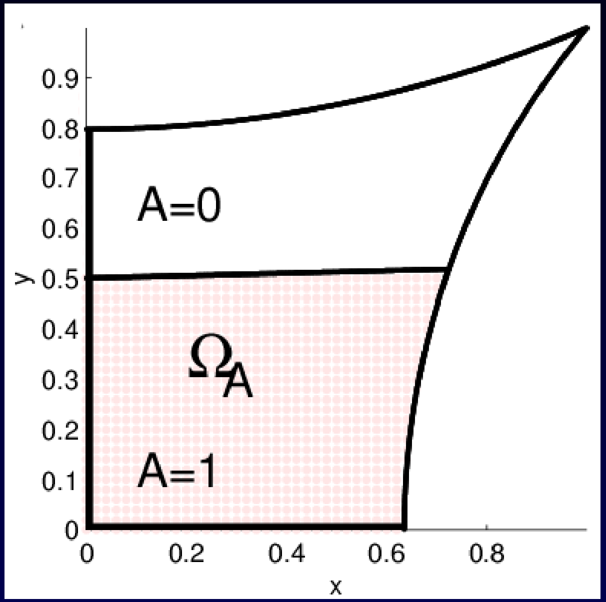
Numerical experiments

(B, Comm. Pure Appl. Math. '06)

dispersing cavity, proven Anosov (Sinai '70)

desymmetrized, generic Λ exponents

test function $A = \begin{cases} 1 & \text{in } \Omega_A, \\ 0 & \text{otherwise} \end{cases}$



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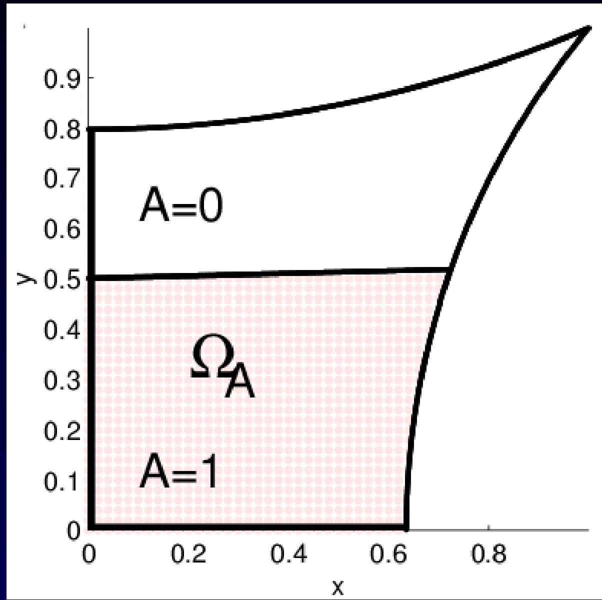


Large-scale study, 30,000 modes in range $j \sim 10^4$ to 10^6, enabled by:

1. Efficient boundary-based numerics for ϕ_j ('scaling method')
2. Matrix elements $\int_{\Omega_A} \phi_j^2 d\mathbf{x}$ via boundary integrals on $\partial\Omega_A$

- 100 times higher in j than any previous studies (*e.g.* Bäcker '98)
- only a few CPU-days total

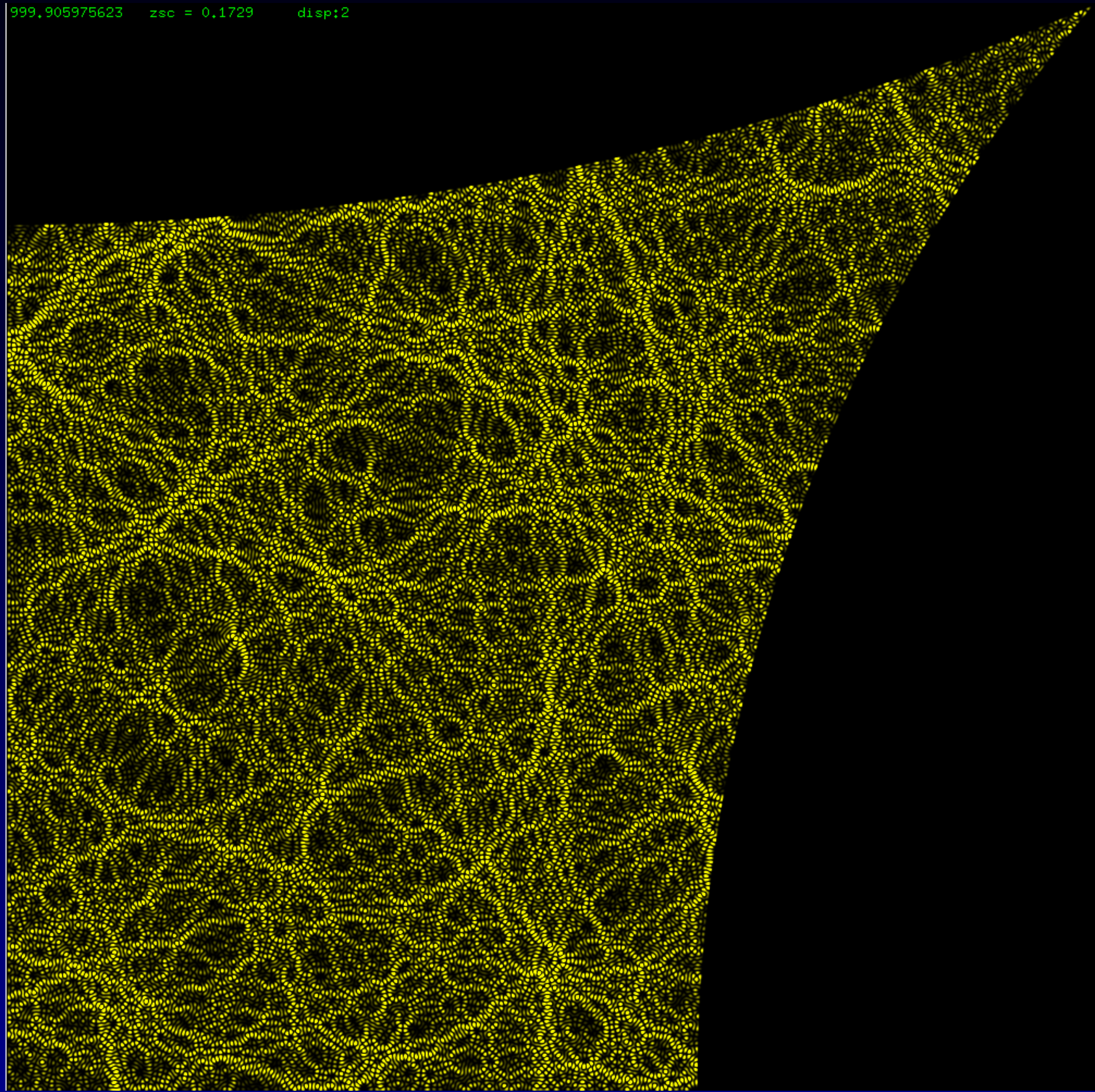
Typical high-frequency ergodic mode

225 wavelengths across system

level number $j \approx 5 \times 10^4$

$E_j \approx 10^6$

Stringiness unexplained…

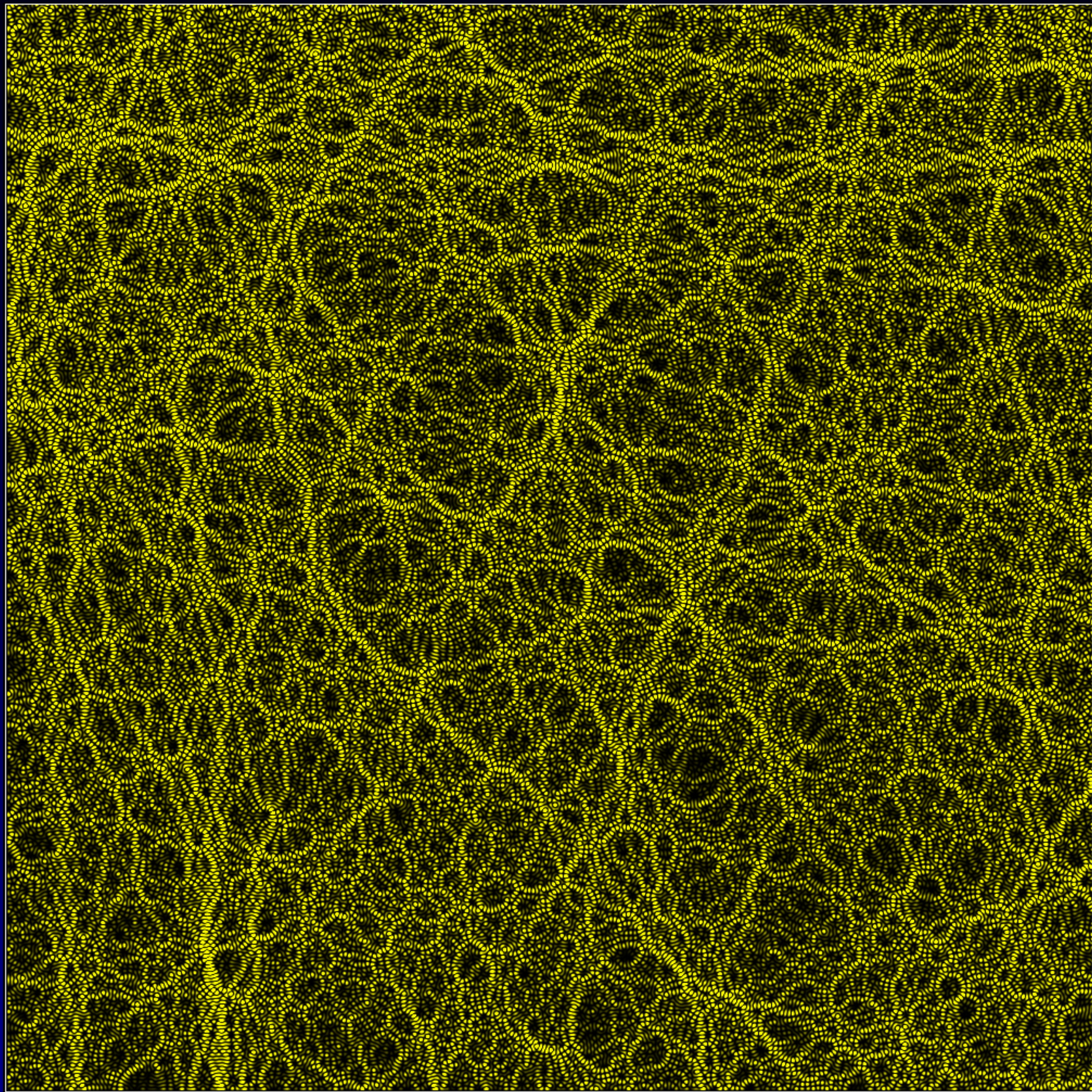


(compare: random sum of plane waves)

$\text{Re} \sum_m a_m e^{i\mathbf{k}_m \cdot \mathbf{x}}$

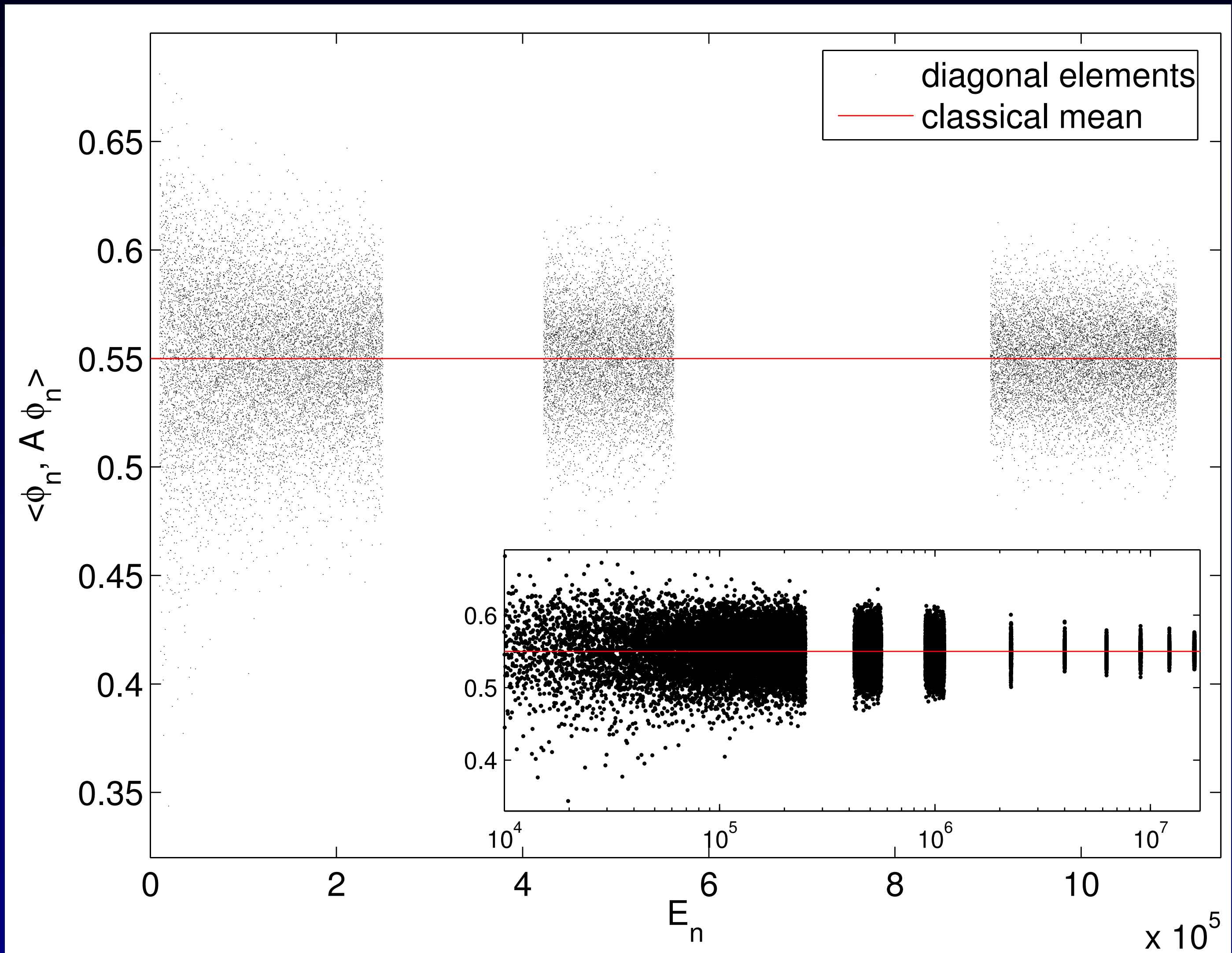
all wavenumbers
$|\mathbf{k}_m| = \sqrt{E} = \text{const.}$

similarly stringy…
interesting
to the eye only?



Raw matrix element data

To reach high E, only use modes in intervals $E_j \in [E, E + L(E)]$



- No outliers $\Rightarrow$ strong evidence for QUE (exceptional density < 3×10^{-5})

At what rate condense to the mean?

interval $I_E = [E, E + L(E)]$ choose width $L(E) = O(E^{1/2})$

eigenvalue count in interval $N(I_E) := \#\{j : E_j \in I_E\}$

'quantum variance' $V_A(E) := \frac{1}{N(I_E)} \sum_{E_j \in I_E} \left| \langle \phi_j, A\phi_j \rangle - \overline{A} \right|^2$

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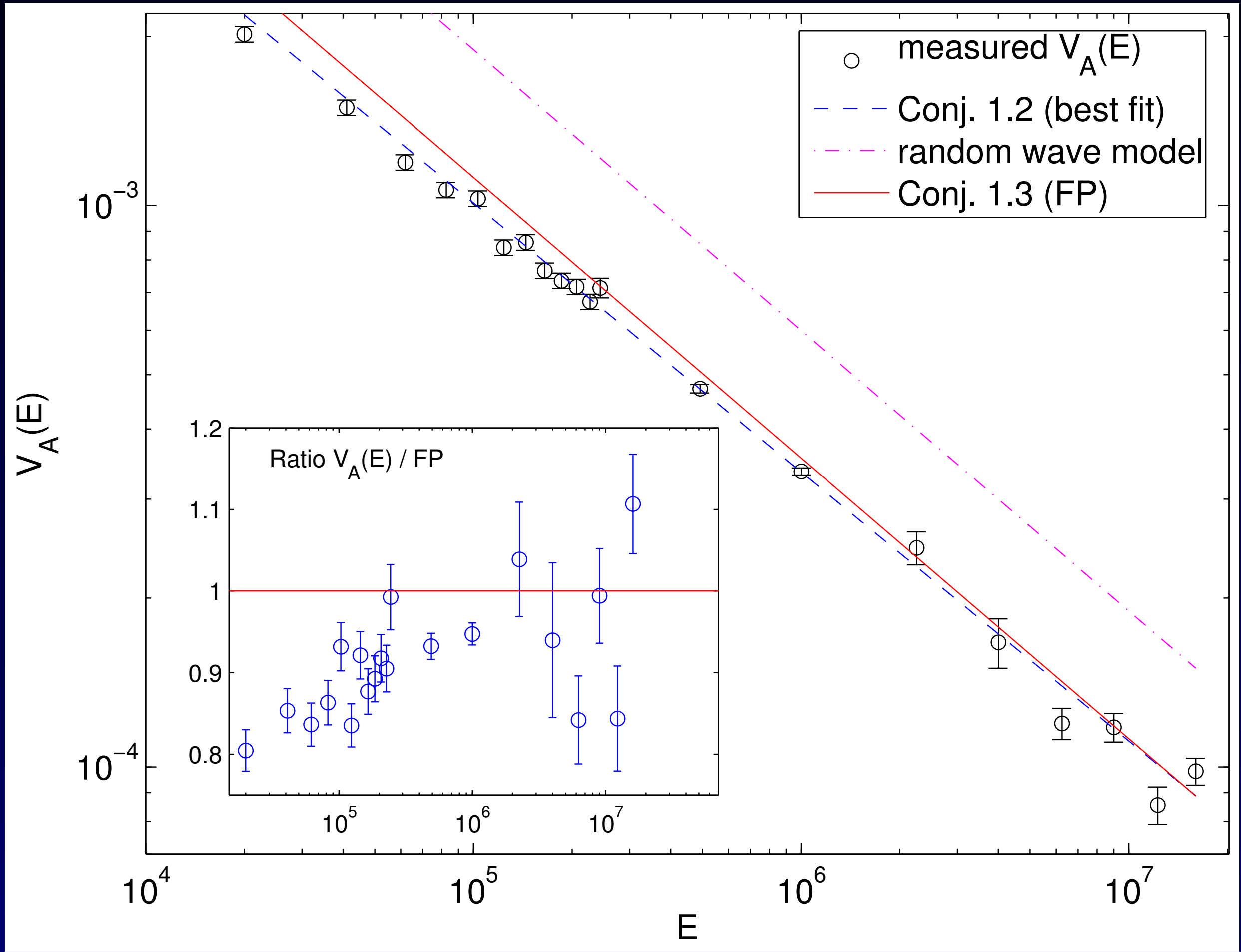
- Conj. (Feingold-Peres '86): $V_A(E) \sim \frac{g\tilde{C}_A(0)}{\text{vol}(\Omega)} E^{-1/2}$

  where $g = 2$ from statistical independence of nearby ϕ_j (heuristic)

  $\tilde{C}_A(\omega) :=$ FT of autocorrelation $\frac{1}{\text{vol}(\Omega)} \int_\Omega A(\mathbf{x}_0)A(\mathbf{x}(\tau))d\mathbf{x} - \overline{A}^2$

Results on quantum ergodicity rate



consistent with power law model $V_A(E) = aE^{-\gamma}$

fit $\gamma = 0.48 \pm 0.01$

very close to $\gamma = 1/2$

prefactor:

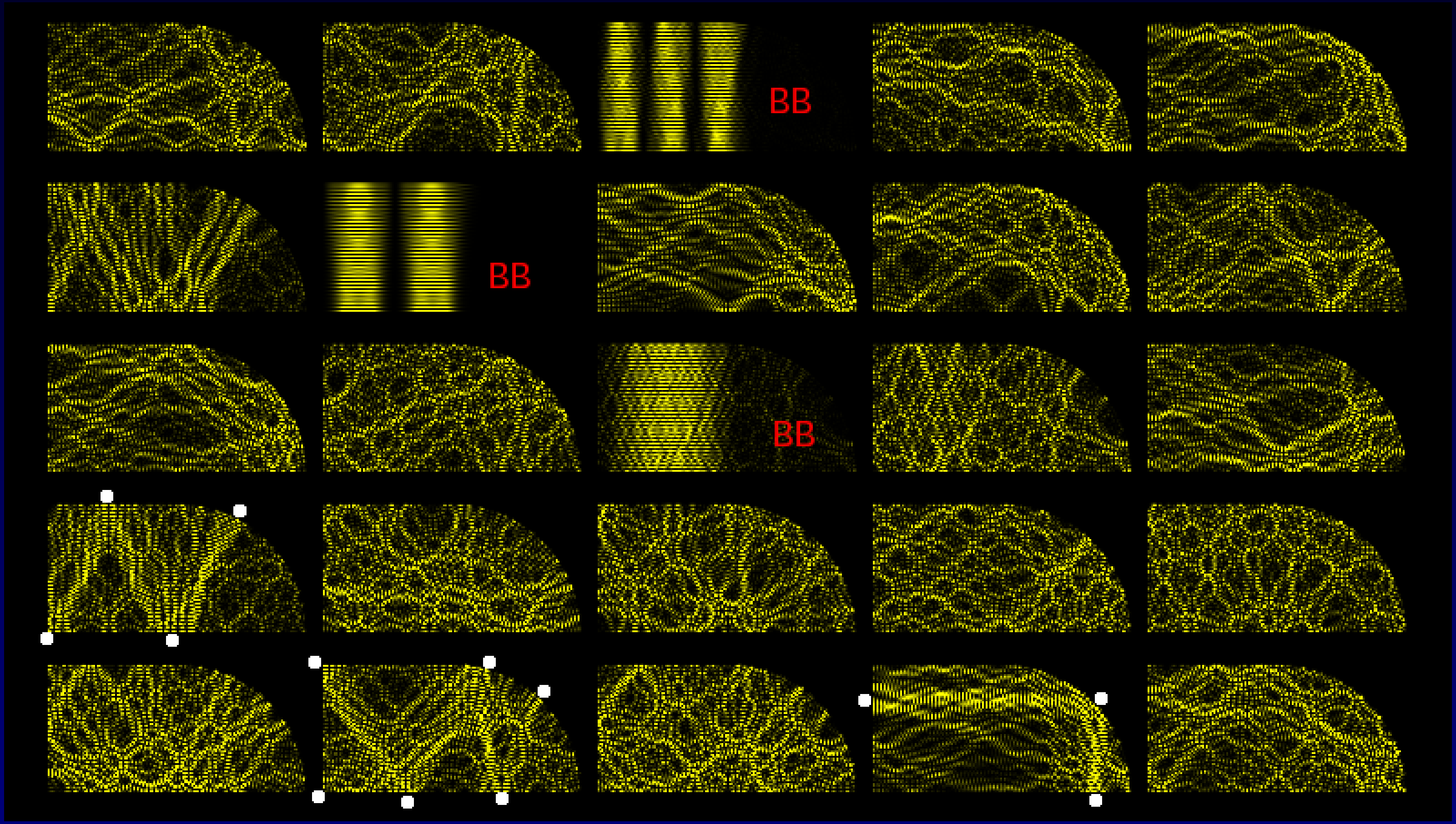
- RW model fails
- FP Conj. succeeds

- large numbers of modes $\longrightarrow$ unprecedented accuracy ($< 1\%$)
- asymptotic regime seen for first time (but more data needed!)
- consistent with FP Conj., convergence very slow: 7% off at $j = 10^5$

III. Scars: shadows of periodic ray orbits

Some high-j modes $|\phi_j|^2$ localize on unstable periodic orbits (UPO)

- discovered by *numerical* study of quarter-stadium modes (Heller '84)



- Note also exceptional 'bouncing ball' BB modes (since not Anosov)
- Apps of scars: dielectric micro-lasers (Tureci et al), tunnel diodes…

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Statistics of modes $\leftrightarrow$ *Greens function* $\leftrightarrow$ *wave propagation*

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$$\sum_{j=1}^{\infty} |\phi_j(\mathbf{x})|^2 \delta(k - k_j) \quad = \quad \frac{2k}{\pi} \lim_{\varepsilon \to 0} \,\mathrm{Im}\, G(\mathbf{x}, \mathbf{x}; k^2 + i\varepsilon)$$

'Local Density Of States' — Greens func for Helmholtz eqn in Ω

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- Then use semiclassical ray theory (heuristic):

$$G(\mathbf{x}, \mathbf{x}; k^2) \approx \big(\text{free space}\big) + \sum_{\substack{\text{returning} \\ \text{orbits } \mathbf{x} \to \mathbf{x}}} \big(\text{geometric factor}\big)\, e^{ik(\text{orbit length})}$$

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Outcome: k-periodic LDOS enhancement along each periodic orbit

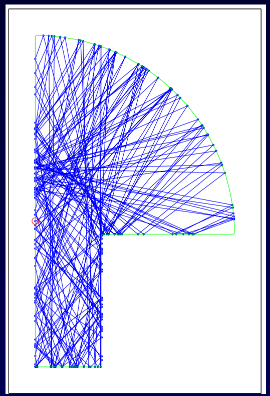
(Gutzwiller, Heller, Bogomolny, Berry, Kaplan, '80-90s)

Mushroom cavity modes

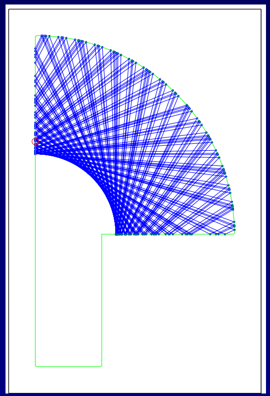
(B-Betcke, `nlin/0611059`)

Unusually simple divided phase space

(Bunimovich '01)



ergodic rays

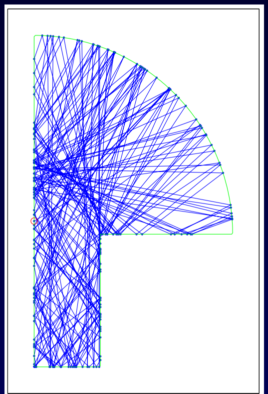


regular rays

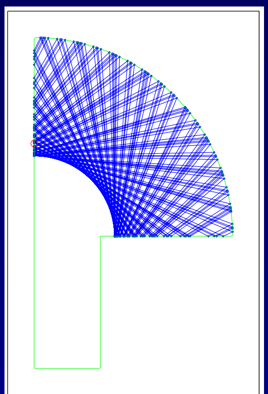
Mushroom cavity modes

(B-Betcke, `nlin/0611059`)

Unusually simple divided phase space (Bunimovich '01)

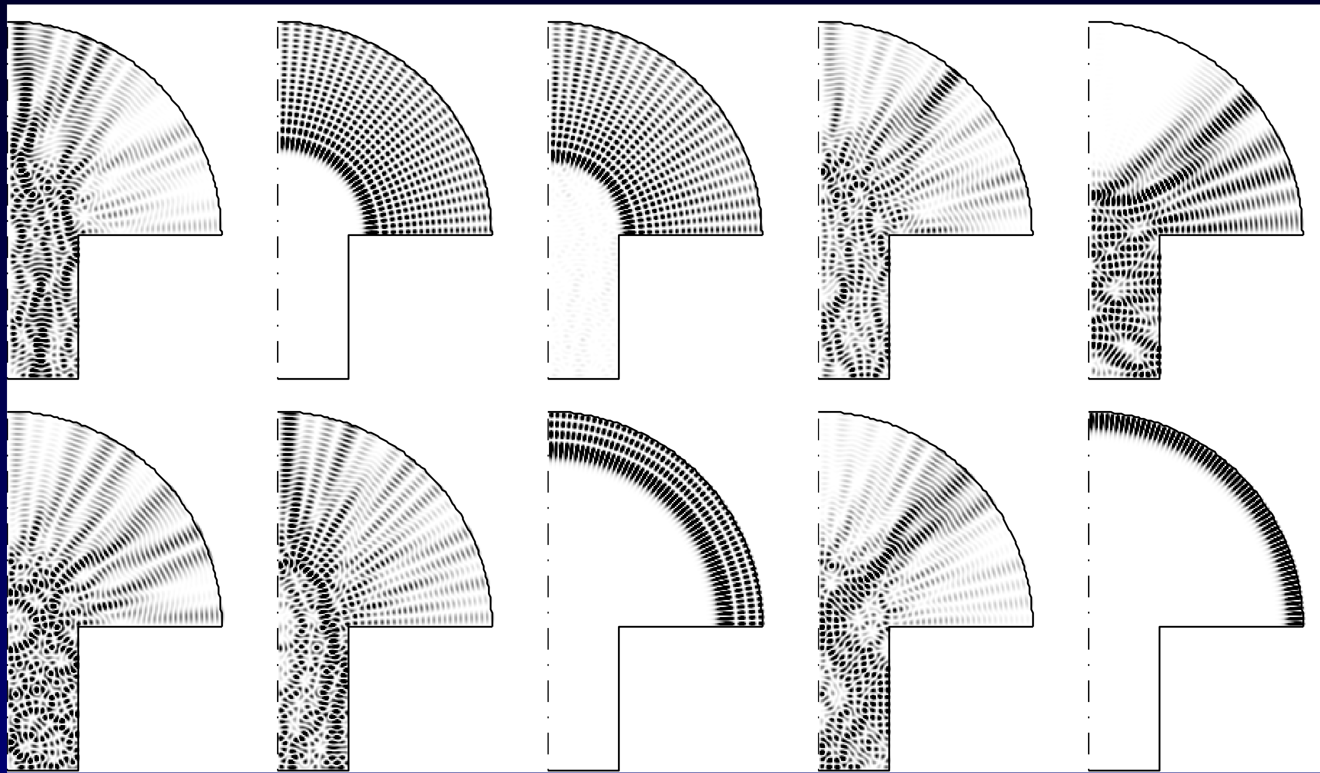


ergodic rays



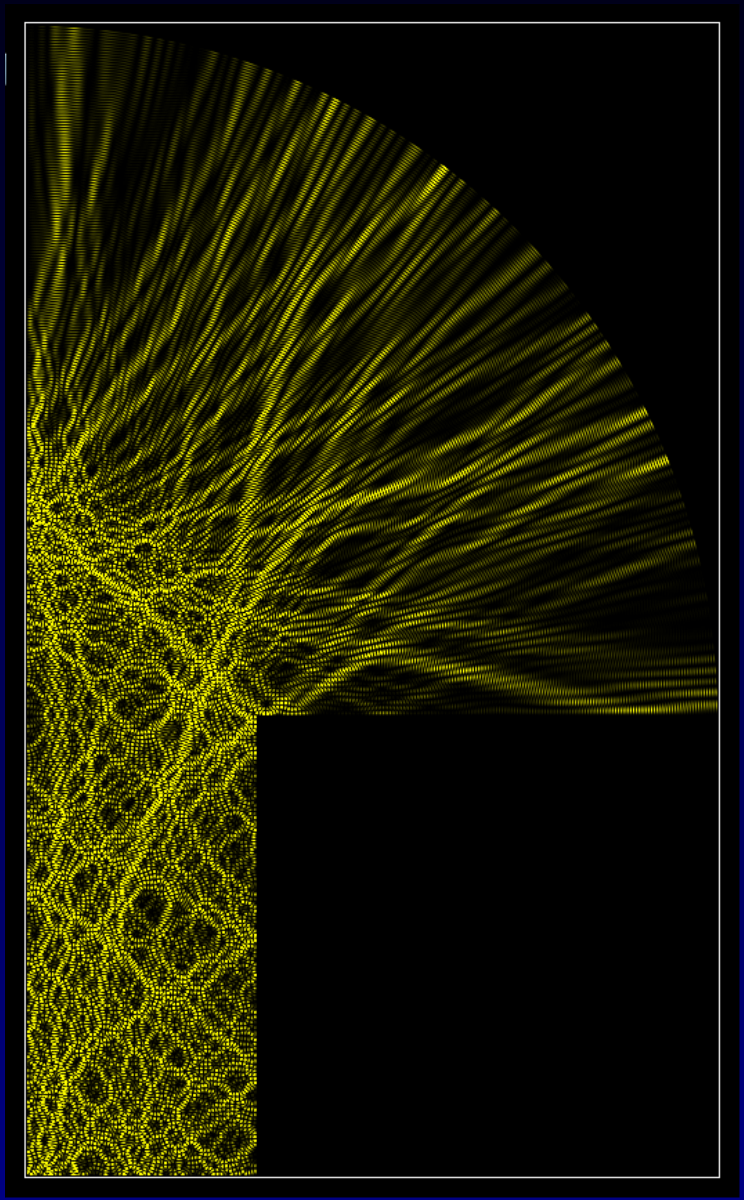
regular rays

First calculation of high-freq modes: $j \approx 2000$

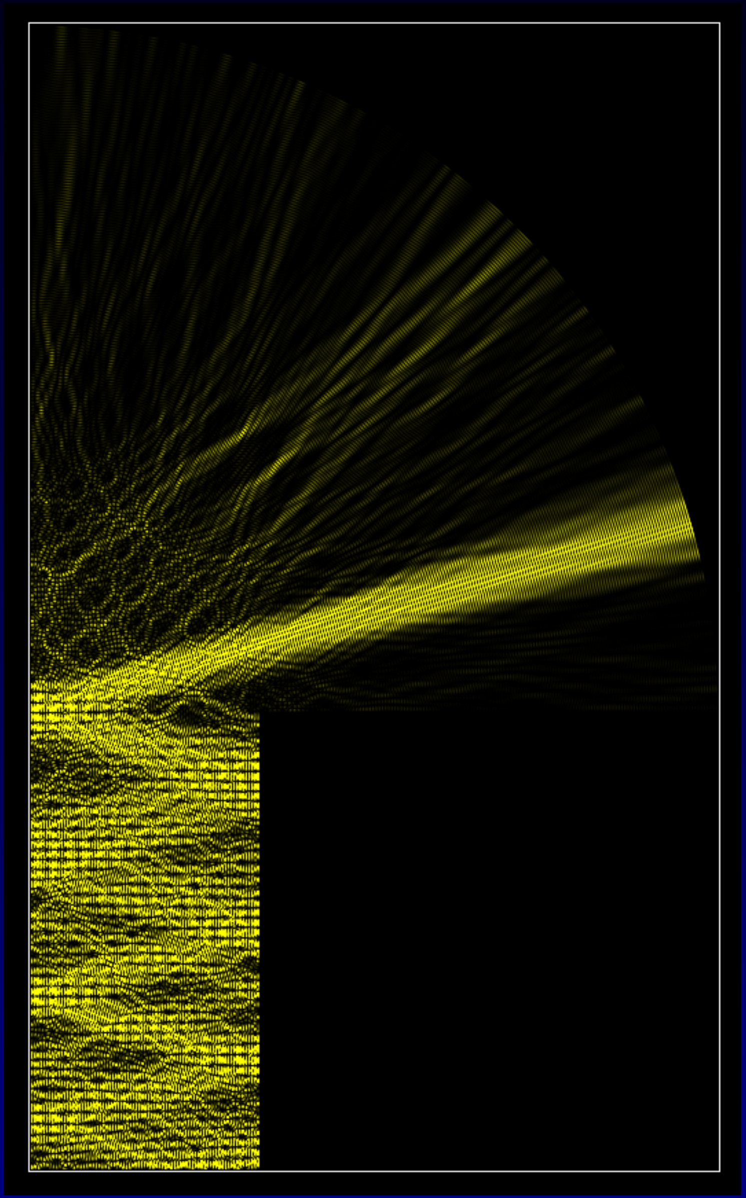


- Percival '73 conjecture verified (Percival '73): modes localize to either regular or chaotic region

Very high freq mushroom modes



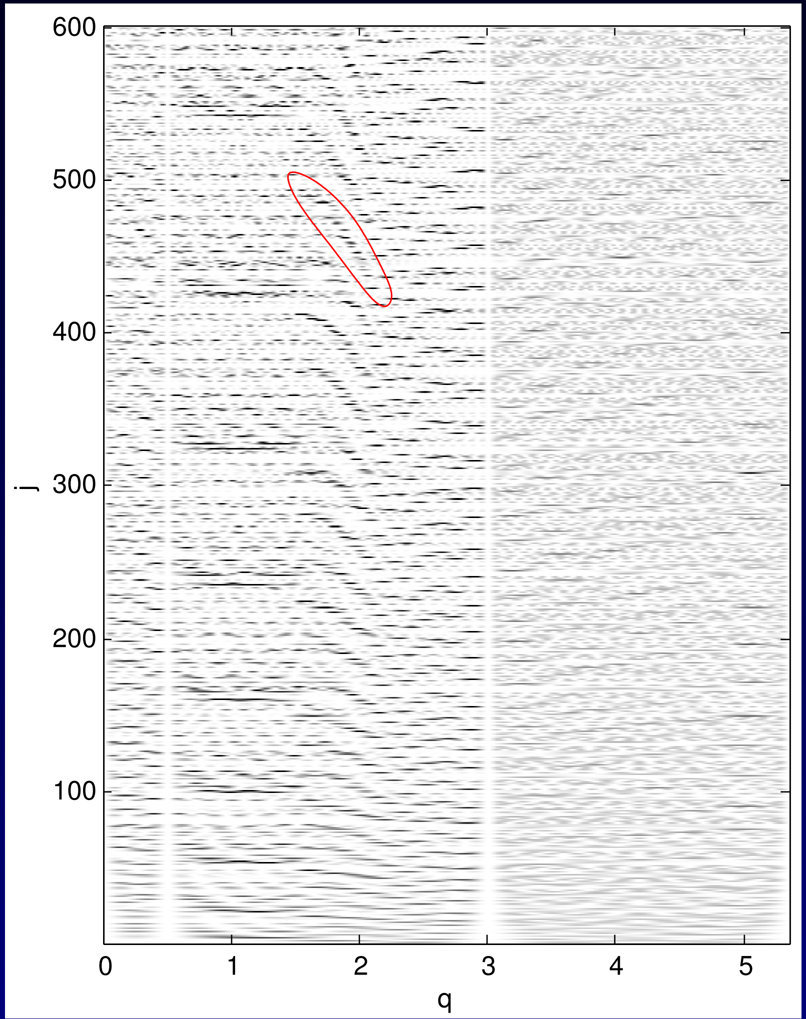
ergodic, equidistributed



ergodic, strongly scarred

New phenomenon: a 'moving scar'

$|\partial_n \phi_j(q)|^2$, boundary location q:

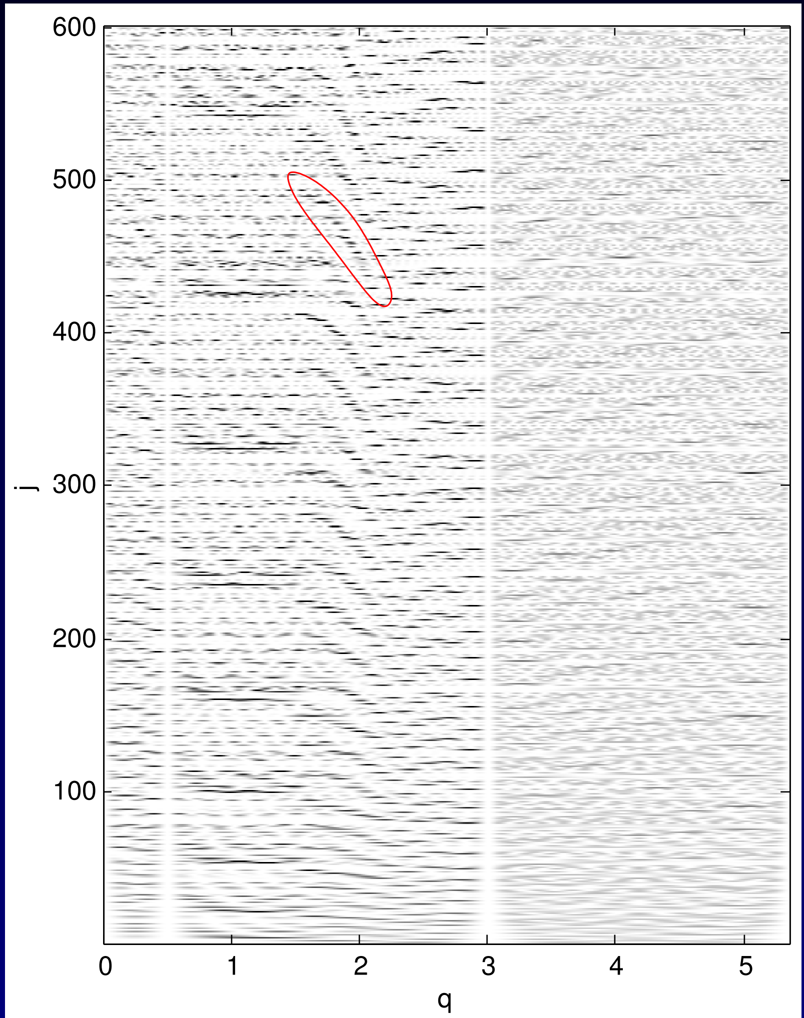


- sloping streak effect

MODES SMOOTHED

New phenomenon: a ‘moving scar’

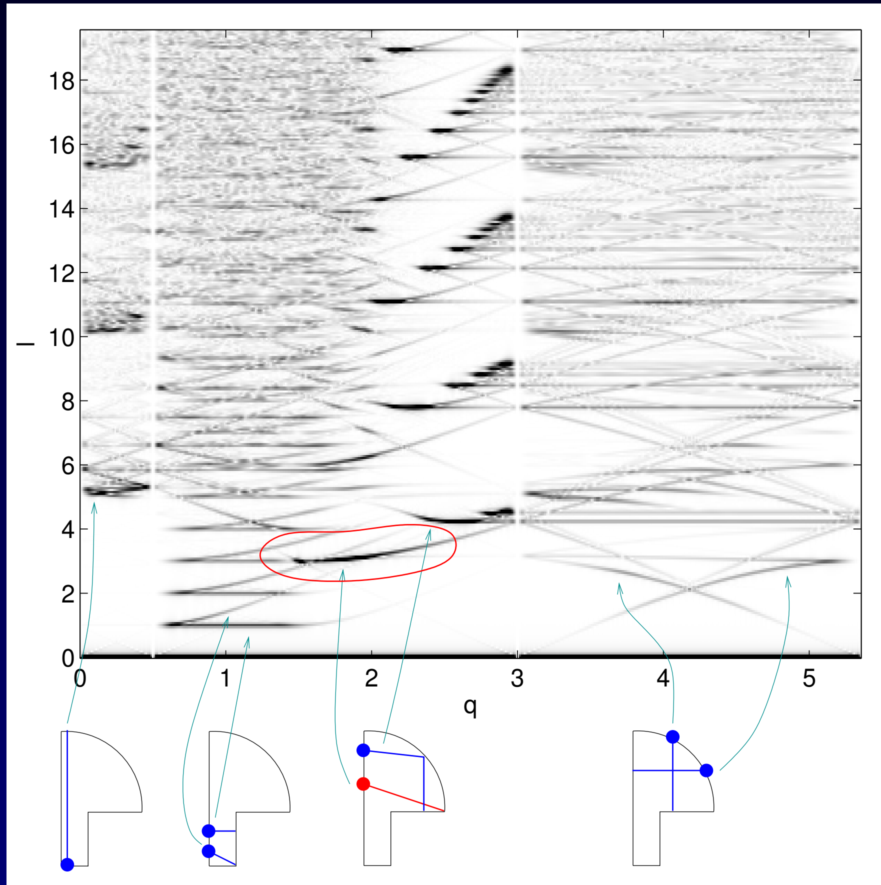
$|\partial_n \phi_j(q)|^2$, boundary location q:



- sloping streak effect

MODES SMOOTHED

take FT along k axis, gives: wave autocorrelation in time



- refocused returning orbits
- return length varies with q

EVOLVE

Conclusion

High frequency asymptotic properties of chaotic modes ϕ_j:

- $|\phi_j(\mathbf{x})|^2$ tends to spatially uniform at conjectured rate
- Scarring on unstable periodic orbits, enhanced by refocusing

Topics not covered:

- Bouncing ball mode leakage (ongoing w/ A. Hassell)
- Quasi-orthogonality of boundary functions $\partial_n \phi_j(s)$

Find out how the numerical method works: Fri am: KOL F 121

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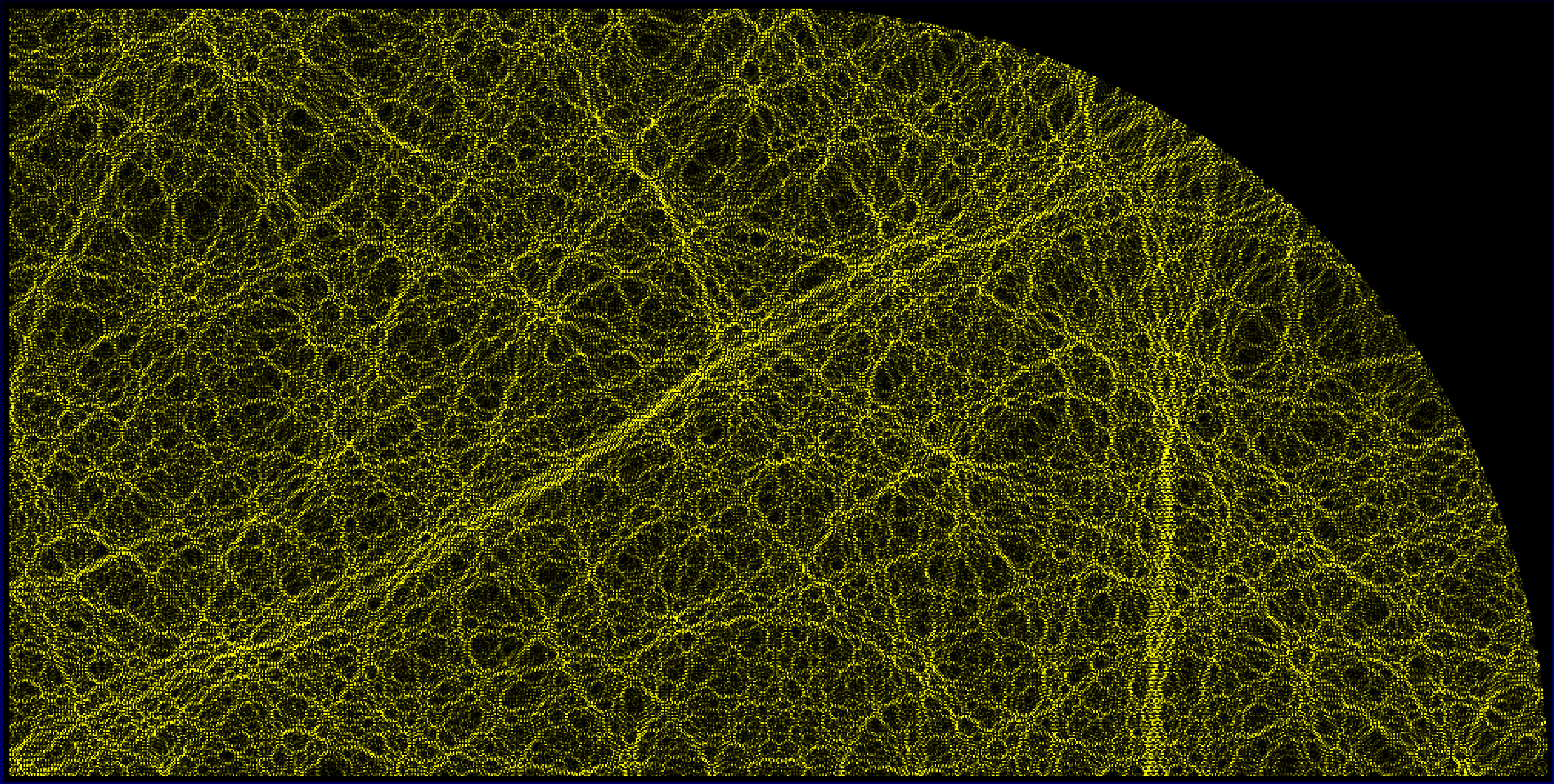
T. Betcke (Manchester)
S. Zelditch (JHU)

Funding: NSF (DMS-0507614)

Preprints, talks, movies:
http://math.dartmouth.edu/∼ahb

made with: Linux, LaTeX, `Prosper`

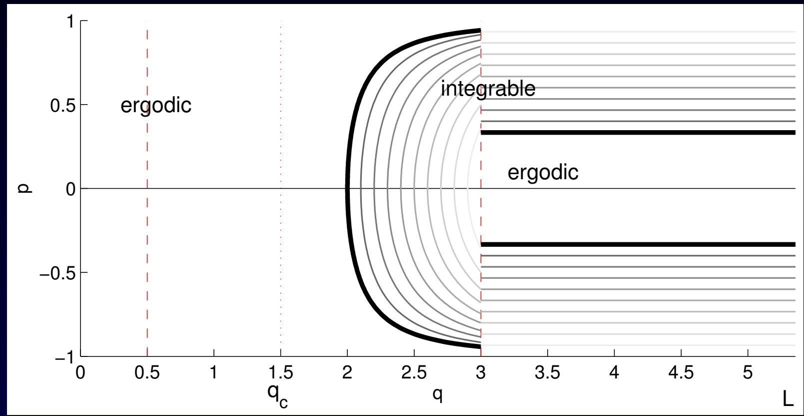
High-eigenvalue quarter-stadium scar

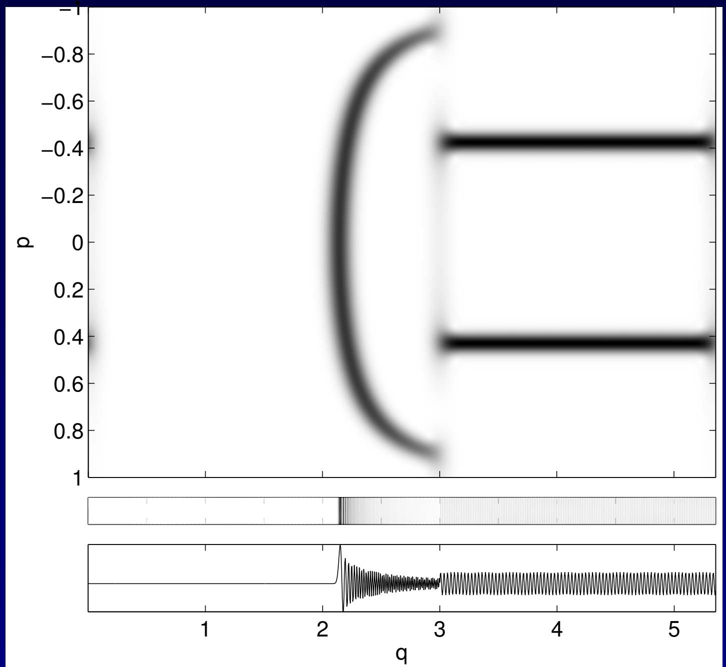


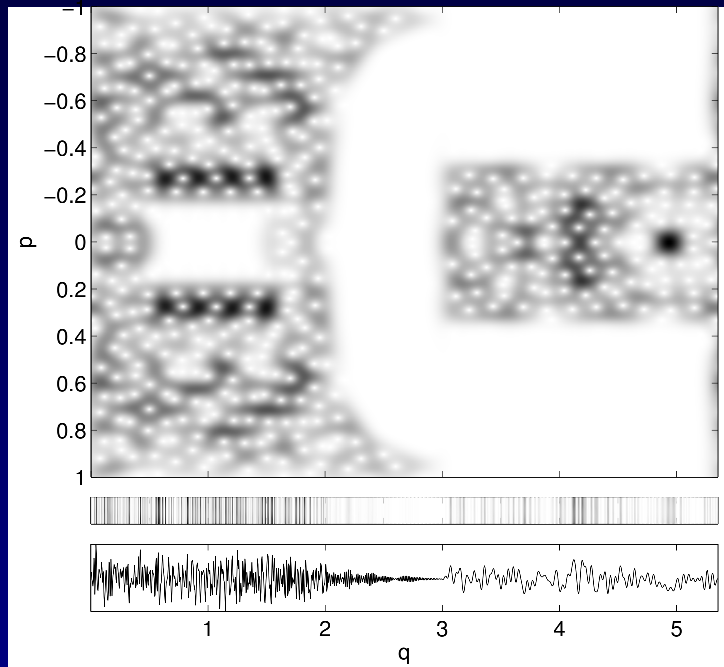
- Our numerical evidence for QUE $\Rightarrow$ scars die, measure $o(1)$

Husimi distributions on mushroom boundary

classical boundary PSOS:







regular

ergodic, strongly scarred

IV. Bouncing Ball modes

(ongoing work w/ A. Hassell)