# Can one “hear” Shape?

## Laplace-Spectra for Shape Recognition

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ICIAM 07

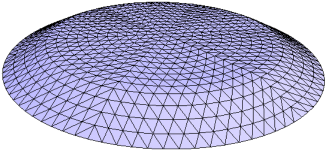
# Overview

# Can one “hear” Shape?



“Can one hear the Shape of a Drum?” (First asked by Bers, then paper by Kac 1966, idea dates back to Weyl 1911)

- The frequencies of a drum depend on its shape.
- This spectrum can be numerically computed if the shape is known.
- E.g., no other shape has the same spectrum as a disk.
- Can the shape be computed from the spectrum?



# Can one “hear” Shape?

## Definition

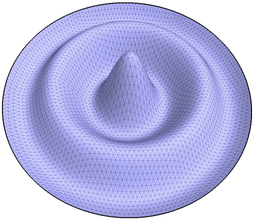
**Helmholtz Equation** (**Laplacian Eigenvalue Problem**):

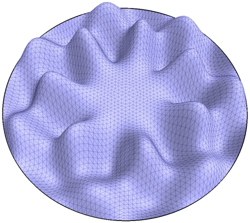
$$\Delta f = -\lambda f, \qquad f : M \to \mathbb{R}$$

Solution: Eigenfunctions $f_i$ with corresponding family of eigenvalues (**Spectrum**):

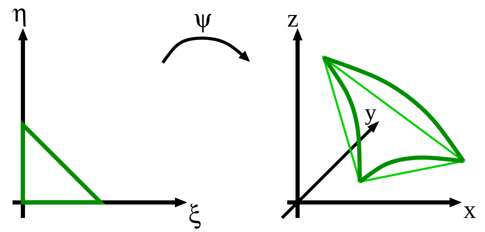
$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty$$

*Here Laplace-Beltrami Operator:* $\Delta f := div(grad\ f)$





# Laplace-Beltrami in Local Coordinates




**Definition (1. fundamental matrix)**

$\psi : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$ be a (local) parametrization of a manifold $M$, then (with $i, j = 1, \ldots, n$ and det the determinant):

$$\begin{aligned} g_{ij} &:= \langle \partial_i \psi, \partial_j \psi \rangle, & G &:= (g_{ij}), \\ W &:= \sqrt{\det G}, & (g^{ij}) &:= G^{-1}. \end{aligned}$$

# Laplace-Beltrami in Local Coordinates



**Definition (Laplace-Beltrami Operator)**

The **Laplace-Beltrami Operator** in local coordinates:

$$\Delta f \quad = \tfrac{1}{W} \sum_{i,j} \partial_i (g^{ij} W \partial_j f)$$

If $M$ is a domain of the Euclidean plane $M \subset \mathbb{R}^2$, the Laplace-Beltrami operator reduces to the well known Laplace operator:

$$\Delta f = \frac{\partial^2 f}{(\partial x)^2} + \frac{\partial^2 f}{(\partial y)^2}$$

# Laplace-Spectrum



Laplace-Beltrami Spectrum for Manifolds with Boundary:

**Dirichlet Boundary Condition**

Function is fixed $f \equiv 0$ on the boundary of $M$

**Neumann Boundary Condition**

Derivative in normal direction is fixed $\frac{\partial f}{\partial n} \equiv 0$ on the boundary of $M$

**Can the shape be computed from the $\lambda_i$?**

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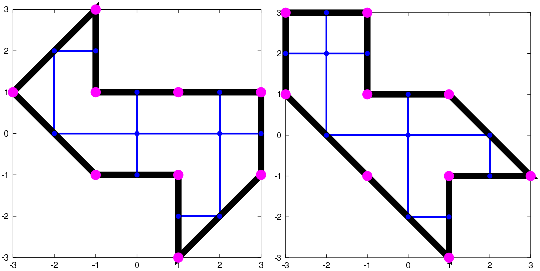
## Answer

**No!** Isospectral drums exist (Gordon, Webb, Wolpert - 1992)

- rare
- concave in 2D
- only pairs




## Geometry

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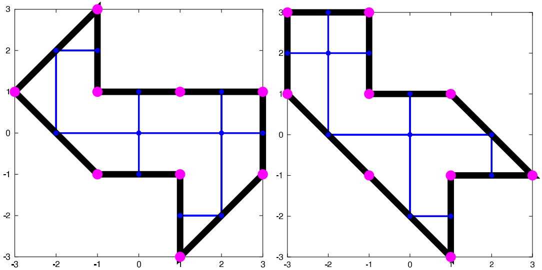

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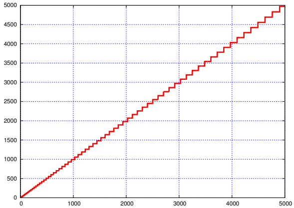
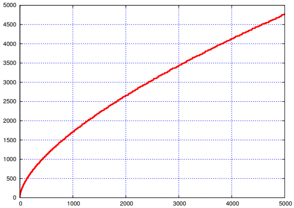
# Weyl's Formular



EW 2D-Sphere




EW 3D-Cube




**Theorem (Weyl - 1911,1912)**

$$\lambda_n \sim \frac{4\pi}{\mathrm{area}(D)}\, n \qquad \text{for } d = 2 \text{ and } n \to \infty$$

$$\lambda_n \sim \left(\frac{6\pi^2}{\mathrm{vol}(D)}\right)^{\frac{2}{3}} n^{\frac{2}{3}} \qquad \text{for } d = 3 \text{ and } n \to \infty.$$

Further geometric and topological information is contained in the Spectrum (Heat-Trace Expansion):

- Riemannian volume
- Riemannian volume of the boundary
- Euler characteristic for closed 2D manifolds
- Number of holes for planar domains

It is possible to extract this data numerically from the beginning sequence of the spectrum (Reuter, Wolter, Peinecke 2006 - first 500 eigenvalues).

# Geometric and Topological Information



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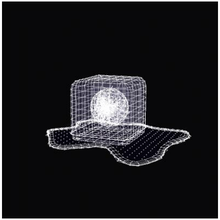
# Comparing and Identifying Shape

Question: What is “Shape” and what is “similar”?

- Is shape just the outer shell of an object (B-Rep)?
- What if the object contains cavities?



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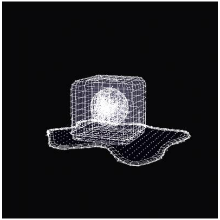

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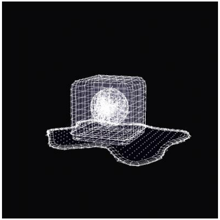

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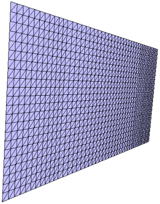
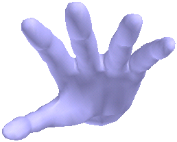
# Comparing and Identifying Shape

- Isometry invariance?





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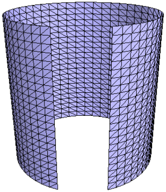
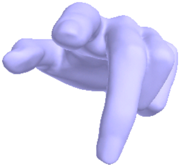
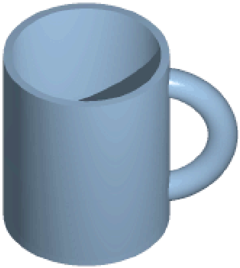




- Homeomorphism invariance? (This goes too far!)



http://en.wikipedia.org/wiki/Topology

# Different Representations and Parameters



Not only do spacial parameters differ, but:

- surfaces and solids can be given in many *different representations* (e.g. parametrized surfaces, 3d polygonal models, implicitly defined surfaces ...).

**Goal of Shape Matching**

To find a method for shape identification and comparison that is

- independent of the given representation of the object.
- invariant w.r.t. congruency, scaling, isometry.

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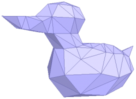
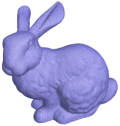
## Shape-Matching





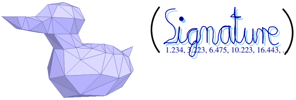
0.) Prior alignment, scaling of the objects: normalization, registration

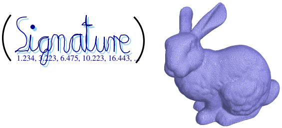
1.) Computation of a simplified representation (Signature, Shape-Descriptor)

2.) Comparison of the signatures, distance computation to measure similarity

# Identification and Comparison

## Shape-Matching

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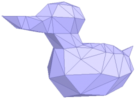



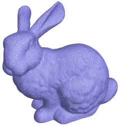

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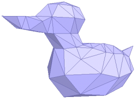
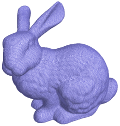



$\longleftrightarrow$

**Distance?**

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Disadvantages of current methods

- Simplification too strong
  (too many objects with identical signatures)
- Missing invariance, complex pre-processing
- Complicated comparison of signatures
  (e.g. graph based signatures)
- Only special representations (Voxels, Triangulations)
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# Overview

# New Signature: Shape-DNA

## (Reuter, Wolter, Peinecke 2005)



We use the (normed) *n*-dim vector of the **smallest** $n$ **eigenvalues** $(\lambda_1, \ldots, \lambda_n)$ of the Laplace operator $\Delta$ as the signature:

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# Variational Formulation - FEM

Multiply Helmholz equation with test functions $\varphi$, then integrate and apply Greens Formula (Variational Form):

$$
\begin{array}{rll}
 & \varphi \Delta f & = -\lambda \varphi f \\
\Leftrightarrow & \iint \varphi \Delta f \; d\sigma & = -\lambda \iint \varphi f \; d\sigma \\
\Leftrightarrow & \iint Df \; G^{-1} \; (D\varphi)^T \; d\sigma & = \lambda \iint \varphi f \; d\sigma
\end{array}
$$

with $Df = (\partial_1 f, \partial_2 f, ...)$.

Approximating $f \approx \sum U_l F_l$ (where $F_l$ form functions):

$$\text{yields:} \quad AU = \lambda BU$$

with the matrices (sparse, symmetric, positiv semi-definit):

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\begin{array}{lll}
A & = (a_{lm}) & := \left( \iint (DF_l) \; G^{-1} \; (DF_m)^T d\sigma \right), \\
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Solve with Lanczos Method from ARPACK

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Solve with Lanczos Method from ARPACK

# Example for the exactness

Param. Sphere
5448 DOF (4s)

0, 2 , 2.0001, 2.0001, 6, 6.000008, 6.000008, 6.0005, 6.0005 ...
$\lambda_{100} = 90.034$ ...
(107.02 linear) ...

Facetted Sphere
11522 DOF (5s)

0, 2.0047, 2.0047, 2.0054, 6.014, 6.014, 6.015, 6.015, 6.016 ...
90.236
(97.884 linear) ...

Exact values

$$\lambda_n = (n-1)n$$

Mulitplicities:
$mult = 2n-1$

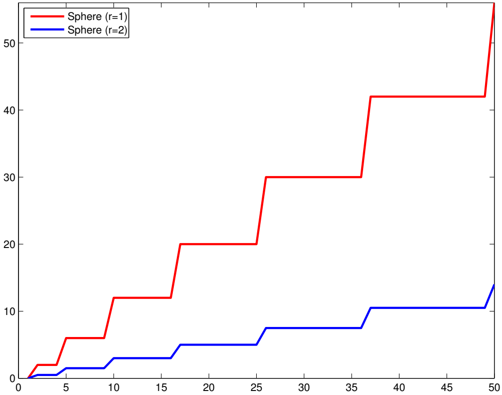
0, 2, 2, 2, 6, 6, 6, 6, 6 ...
90 ...

# Dilation



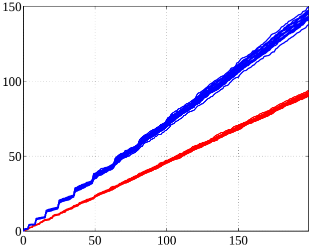
## Dilation

If $M$ is scaled by $s$, spectrum is scaled by $s^{-2}$ (in any dimension).

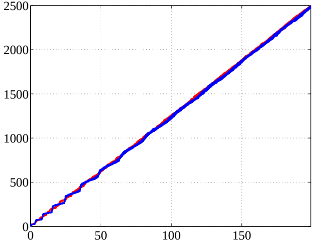


# Normalization

Two classes of spectra of spheres and ellipsoids with noise
blue : noisy spheres , red : noisy ellipsoids




*unnormalized*

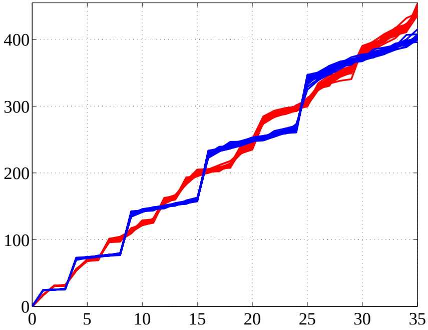



*area normalized*

- Shape analysis results depend on chosen normalization
- Unnormalized: Mainly differences in area/volume

# Normalization

Zoom-ins on the two aligned spectra classes:



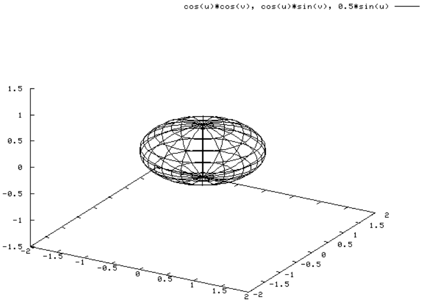

blue : noisy spheres , red : noisy ellipsoids

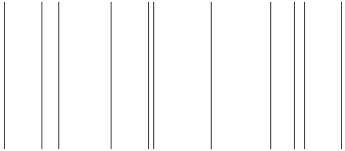
- Area/volume normalization shows if additional **shape differences** exist.

# Continuous Dependency on Deformation



The spectrum depends **continuously** on the shape.



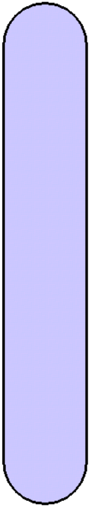



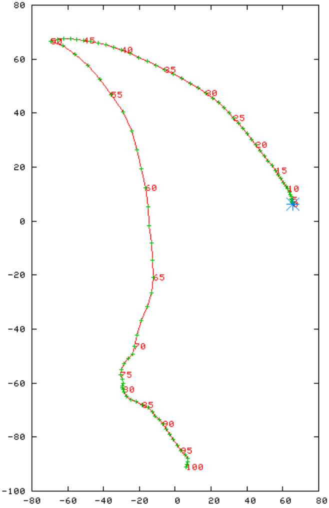
# MDS Plot 2D - Medial Bar Deformation



## Continuous Dependency on Deformation

M.Reuter

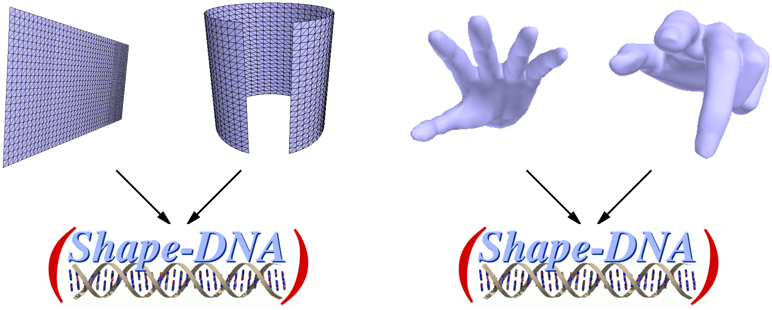




# Isometry Invariance



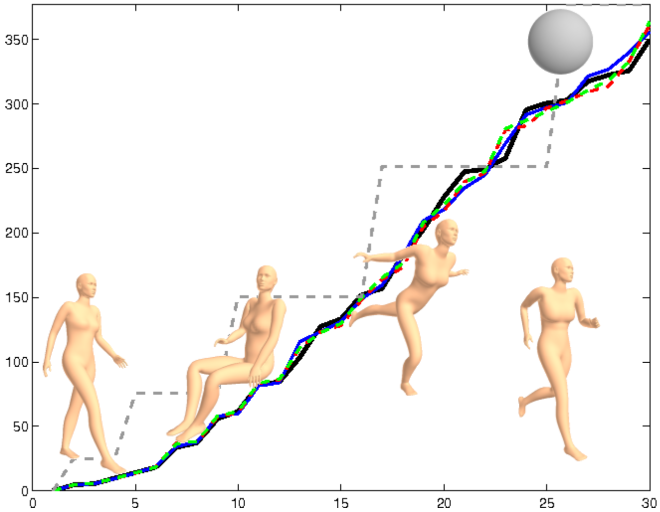
Isometric objects have the same spectrum!




- Spectrum is independent of object's spacial position.

# Isometry Invariance






(Models courtesy of Miralab)

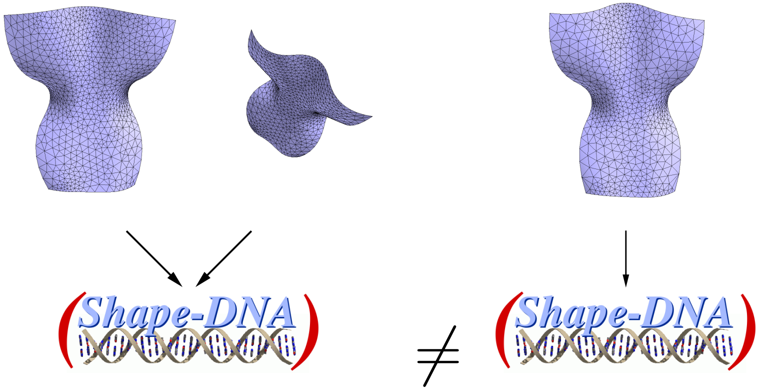
# Overview

# Identification in DB, Copyright protection, Quality assessment



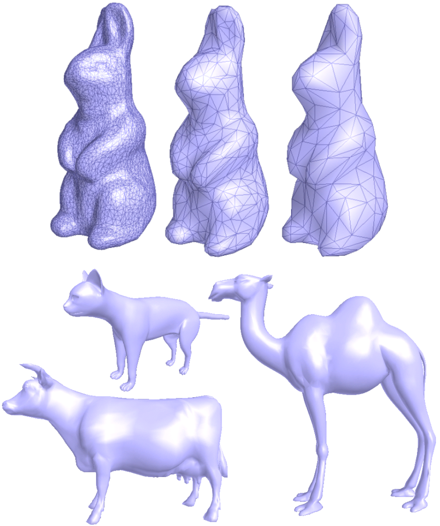
Different representations ⇒

- challenging to identify a protected object
- challenging to retrieve a specific object from DB



# Triangulation of Deformed Spheres





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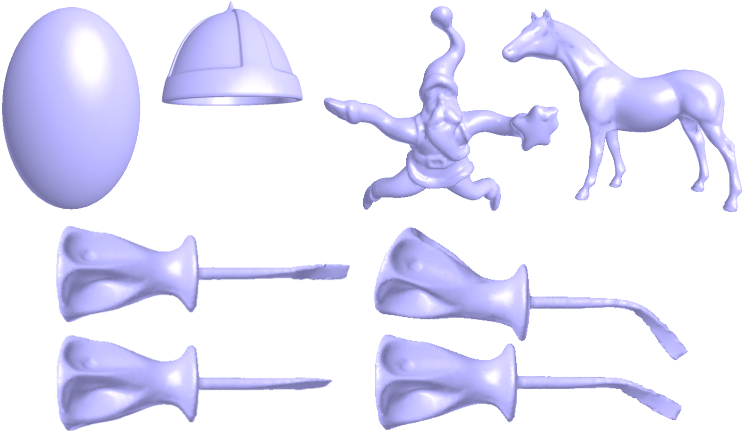
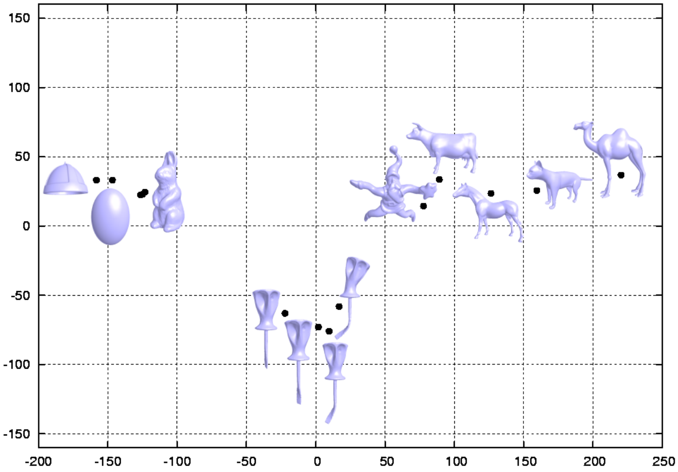


# MDS Plot 2D - Deformed Spheres

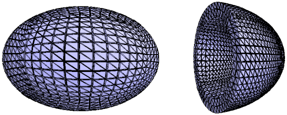


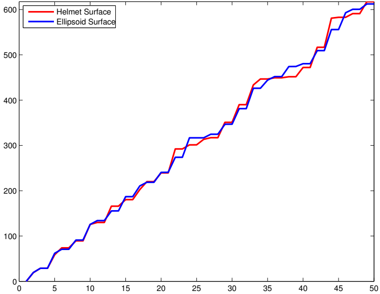


# Deformed Spheres in 3D

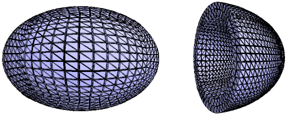


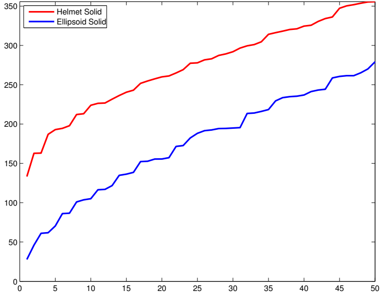
## 2D Surface ShapeDNA






## 3D Solid ShapeDNA






For solid bodies in $\mathbb{R}^3$ isometry is equivalent to congruency.

Martin Reuter



## Global Shape Analysis on caudate nucleus (Brain MRI)

Populations:

**SPD**

32 female subjects diagnosed with Schizotypal Personality Disorder (SPD)

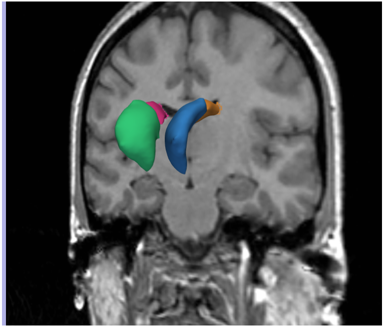
**NC**

29 female normal control (NC) subjects

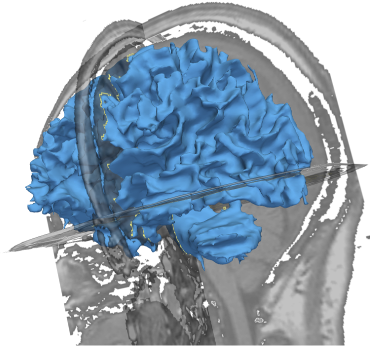
(Harvard Medical - Psychiatry NeuroImaging Laboratory)

# Rendering of the Caudate Nucleus





Coronal view.

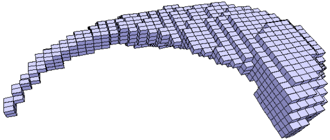


Involved in memory function, emotion processing, and learning.

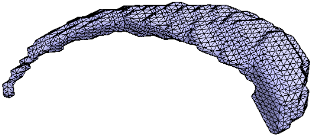
The caudate nucleus was delineated manually by an expert.

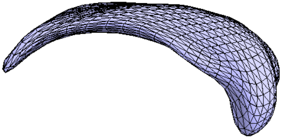
# Iso Surfaces from MRI Data

Shape comparison either on volumetric data (e.g. tetrahedrization or directly on binary voxel data):



or extraction of (smoothed) iso surfaces:





# Statistical Analysis



Permutation tests to compare two populations (200,000 permutations)

- Unnormalized shapes show statistically significant differences (expected: volume, area differences).
- Stat. sign. differences with normalized shapeDNA indicate true shape differences.
- For 3D voxels Neumann spectra indicate differences in smaller features.

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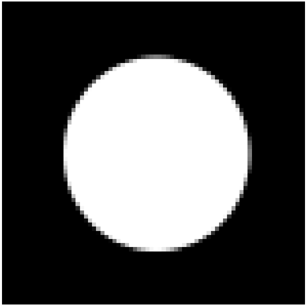
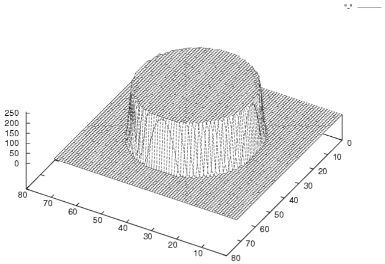
# Images (Peinecke, Wolter, Reuter 2007)

Height function:






or

**Mass-Density Function**

$\Delta f = -\lambda \rho f$ with the mass-density function $\rho$.

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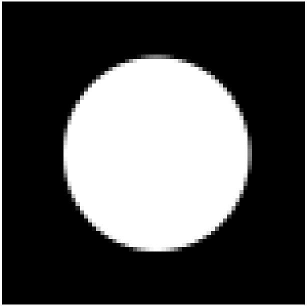

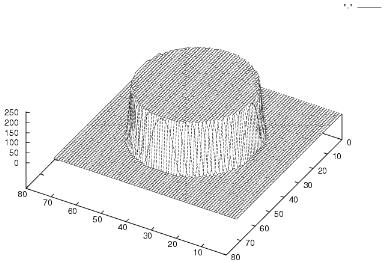


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# Vectorfields



- We have seen examples in scalar fields (MRI data, images)
- Extension to vector fields $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m > 1$?
- $f$ generally not a parametrization of a manifold
- extending the $m$ coordinates of the function $f$ with the $n$ parameter values:

$$F(x_1, ..., x_n) = (x_1, ..., x_n, f_1, ..., f_m)$$

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# Conclusion



- ShapeDNA has many desired properties for shape matching
- Mainly: isometry invariance
- Can be computed very accurately with FEM
- Volumetric spectra are feasible for 3D shape analysis
- Method universally applicable for imaging and CAD applications
- Comparison of shape based on feature size (frequency of eigenfunctions)

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# Thanks



Thank you very much for your attention !

Publications can be found at
http://reuter.mit.edu