Can one “hear” Shape?
Laplace-Spectra for Shape Recognition

Dr. Martin Reuter

Department of Mechanical Engineering
Massachusetts Institute of Technology

ICIAM 07
Overview

1. Shape
   - Hearing Shape
   - Comparing and Identifying Shape Signatures

2. Shape-DNA
   - Laplace-Spectrum as a Signature
   - Implementation
   - Properties of the Spectrum

3. Applications
   - Identification and Similarity Detection
   - Global Analysis of Medical Data
Can one “hear” Shape?

“Can one hear the Shape of a Drum?” (First asked by Bers, then paper by Kac 1966, idea dates back to Weyl 1911)

- The frequencies of a drum depend on its shape.
- This spectrum can be numerically computed if the shape is known.
- E.g., no other shape has the same spectrum as a disk.
- Can the shape be computed from the spectrum?
Can one “hear” Shape?

**Definition**

**Helmholtz Equation (Laplacian Eigenvalue Problem):**

\[ \Delta f = -\lambda f, \quad f : M \to \mathbb{R} \]

Solution: Eigenfunctions \( f_i \) with corresponding family of eigenvalues (**Spectrum**):

\[ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty \]

*Here Laplace-Beltrami Operator: \( \Delta f := \text{div}(\text{grad } f) \)*
Laplace-Beltrami in Local Coordinates

**Definition (1. fundamental matrix)**

\[ \psi : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k} \] be a (local) parametrization of a manifold \( M \), then (with \( i, j = 1, \ldots, n \) and \( \det \) the determinant):

\[
\begin{align*}
g_{ij} &:= \langle \partial_i \psi, \partial_j \psi \rangle, \quad G := (g_{ij}), \\
W &:= \sqrt{\det G}, \quad (g^{ij}) := G^{-1}.
\end{align*}
\]
Laplace-Beltrami in Local Coordinates

**Definition (Laplace-Beltrami Operator)**

The *Laplace-Beltrami Operator* in local coordinates:

\[
\Delta f = \frac{1}{W} \sum_{i,j} \partial_i (g^{ij} W \partial_j f)
\]

If \( M \) is a domain of the Euclidean plane \( M \subset \mathbb{R}^2 \), the Laplace-Beltrami operator reduces to the well known Laplace operator:

\[
\Delta f = \frac{\partial^2 f}{(\partial x)^2} + \frac{\partial^2 f}{(\partial y)^2}
\]
Laplace-Beltrami Spectrum for Manifolds with Boundary:

**Dirichlet Boundary Condition**

Function is fixed $f \equiv 0$ on the boundary of $M$

**Neumann Boundary Condition**

Derivative in normal direction is fixed $\frac{\partial f}{\partial n} \equiv 0$ on the boundary of $M$

Can the shape be computed from the $\lambda_i$?
Laplace-Beltrami Spectrum for Manifolds with Boundary:

**Dirichlet Boundary Condition**
Function is fixed $f \equiv 0$ on the boundary of $M$

**Neumann Boundary Condition**
Derivative in normal direction is fixed $\frac{\partial f}{\partial n} \equiv 0$ on the boundary of $M$

Can the shape be computed from the $\lambda_i$?
Laplace-Beltrami Spectrum for Manifolds with Boundary:

**Dirichlet Boundary Condition**
Function is fixed $f \equiv 0$ on the boundary of $M$

**Neumann Boundary Condition**
Derivative in normal direction is fixed $\frac{\partial f}{\partial n} \equiv 0$ on the boundary of $M$

Can the shape be computed from the $\lambda_i$?
Can one “hear” Shape?

**Answer**

No! Isospectral drums exist (Gordon, Webb, Wolpert - 1992)

- rare
- concave in 2D
- only pairs

Geometry

Nevertheless, they share area, boundary length, genus...
Can one “hear” Shape?

**Answer**

No! Isospectral drums exist (Gordon, Webb, Wolpert - 1992)

- rare
- concave in 2D
- only pairs

Nevertheless, they share area, boundary length, genus...

**Construction**

- Geometry
  - Laplace-Spectrum as a Signature
  - Implementation
  - Properties of the Spectrum

**Applications**

- Identification and Similarity Detection
- Global Analysis of Medical Data
Theorem (Weyl - 1911,1912)

\[
\lambda_n \sim \frac{4\pi}{\text{area}(D)} n \quad \text{for } d = 2 \text{ and } n \to \infty
\]

\[
\lambda_n \sim \left( \frac{6\pi^2}{\text{vol}(D)} \right)^{\frac{2}{3}} n^{\frac{2}{3}} \quad \text{for } d = 3 \text{ and } n \to \infty.
\]
Further geometric and topological information is contained in the Spectrum (Heat-Trace Expansion):

- Riemannian volume
- Riemannian volume of the boundary
- Euler characteristic for closed 2D manifolds
- Number of holes for planar domains

It is possible to extract this data numerically from the beginning sequence of the spectrum (Reuter, Wolter, Peinecke 2006 - first 500 eigenvalues).
Further geometric and topological information is contained in the Spectrum (Heat-Trace Expansion):

- Riemannian volume
- Riemannian volume of the boundary
- Euler characteristic for closed 2D manifolds
- Number of holes for planar domains

It is possible to extract this data numerically from the beginning sequence of the spectrum (Reuter, Wolter, Peinecke 2006 - first 500 eigenvalues).
Comparing and Identifying Shape

Question: What is “Shape” and what is “similar”?

- Is shape just the outer shell of an object (B-Rep)?
- What if the object contains cavities?

- Shape should be invariant wrt translation and rotation (congruence)!
- How about scaling invariance (sometimes)?
Comparing and Identifying Shape

Question: What is “Shape” and what is “similar”?  
- Is shape just the outer shell of an object (B-Rep)?
- What if the object contains cavities?

Shape should be invariant wrt translation and rotation (congruence)!
- How about scaling invariance (sometimes)?
Question: What is “Shape” and what is “similar”?

- Is shape just the outer shell of an object (B-Rep)?
- What if the object contains cavities?

- Shape should be invariant wrt translation and rotation (congruence)!
- How about scaling invariance (sometimes)?
Comparing and Identifying Shape

Industry invariance?
Homeomorphism invariance? (This goes too far!)

http://en.wikipedia.org/wiki/Topology
Not only do spacial parameters differ, but:

- surfaces and solids can be given in many *different representations* (e.g. parametrized surfaces, 3d polygonal models, implicitly defined surfaces ...).

**Goal of Shape Matching**

To find a method for shape identification and comparison that is

- independent of the given representation of the object.
- invariant w.r.t. congruency, scaling, isometry.
Not only do spacial parameters differ, but:

- surfaces and solids can be given in many different representations (e.g. parametrized surfaces, 3d polygonal models, implicitly defined surfaces ...).

**Goal of Shape Matching**

To find a method for shape identification and comparison that is

- independent of the given representation of the object.
- invariant w.r.t. congruency, scaling, isometry.
Identification and Comparison

Shape-Matching

0.) Prior alignment, scaling of the objects: normalization, registration
1.) Computation of a simplified representation (Signature, Shape-Descriptor)
2.) Comparison of the signatures, distance computation to measure similarity
Identification and Comparison

Shape-Matching

0.) Prior alignment, scaling of the objects: normalization, registration

1.) Computation of a simplified representation (Signature, Shape-Descriptor)

2.) Comparison of the signatures, distance computation to measure similarity
Identification and Comparison

Shape-Matching

0.) Prior alignment, scaling of the objects: normalization, registration

1.) Computation of a simplified representation (Signature, Shape-Descriptor)

2.) Comparison of the signatures, distance computation to measure similarity
Shape-Matching

0.) Prior alignment, scaling of the objects: normalization, registration

1.) Computation of a simplified representation (Signature, Shape-Descriptor)

2.) Comparison of the signatures, distance computation to measure similarity
Disadvantages of current methods

- Simplification too strong (too many objects with identical signatures)
- Missing invariance, complex pre-processing
- Complicated comparison of signatures (e.g. graph based signatures)
- Only special representations (Voxels, Triangulations)
- Depending on supplementary information / context
Disadvantages of current methods

- Simplification too strong
  (too many objects with identical signatures)
- Missing invariance, complex pre-processing
- Complicated comparison of signatures
  (e.g. graph based signatures)
- Only special representations (Voxels, Triangulations)
- Depending on supplementary information / context
Disadvantages of current methods

- Simplification too strong
  (too many objects with identical signatures)
- Missing invariance, complex pre-processing
- Complicated comparison of signatures
  (e.g. graph based signatures)
- Only special representations (Voxels, Triangulations)
- Depending on supplementary information / context
Disadvantages of current methods

- Simplification too strong
  (too many objects with identical signatures)
- Missing invariance, complex pre-processing
- Complicated comparison of signatures
  (e.g. graph based signatures)
- Only special representations (Voxels, Triangulations)
- Depending on supplementary information / context
Disadvantages of current methods

- Simplification too strong
  (too many objects with identical signatures)
- Missing invariance, complex pre-processing
- Complicated comparison of signatures
  (e.g. graph based signatures)
- Only special representations (Voxels, Triangulations)
- Depending on supplementary information / context
Overview

1. Shape
   - Hearing Shape
   - Comparing and Identifying Shape Signatures

2. Shape-DNA
   - Laplace-Spectrum as a Signature
   - Implementation
   - Properties of the Spectrum

3. Applications
   - Identification and Similarity Detection
   - Global Analysis of Medical Data
New Signature: Shape-DNA
(Reuter, Wolter, Peinecke 2005)

We use the (normed) \( n \)-dim vector of the smallest \( n \) eigenvalues \((\lambda_1, \ldots, \lambda_n)\) of the Laplace operator \( \Delta \) as the signature:

- Invariant wrt translation, rotation and (where required) scaling
- No registration necessary
- Surfaces & solids (even with cavities), arbitrary genus
- Independent of representation
- Isometry invariant
- Simple distance computation of the signatures
- No user interaction
We use the (normed) $n$-dim vector of the **smallest** $n$ eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of the Laplace operator $\Delta$ as the signature:

- Invariant wrt translation, rotation and (where required) scaling
- No registration necessary
- Surfaces & solids (even with cavities), arbitrary genus
- Independent of representation
- Isometry invariant
- Simple distance computation of the signatures
- No user interaction
We use the (normed) $n$-dim vector of the smallest $n$ eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of the Laplace operator $\Delta$ as the signature:

- Invariant wrt translation, rotation and (where required) scaling
- No registration necessary
- Surfaces & solids (even with cavities), arbitrary genus
- Independent of representation
- Isometry invariant
- Simple distance computation of the signatures
- No user interaction
New Signature: Shape-DNA
(Reuter, Wolter, Peinecke 2005)

We use the (normed) \( n \)-dim vector of the smallest \( n \) eigenvalues \((\lambda_1, \ldots, \lambda_n)\) of the Laplace operator \( \Delta \) as the signature:

- Invariant wrt translation, rotation and (where required) scaling
- No registration necessary
- Surfaces & solids (even with cavities), arbitrary genus
- Independent of representation
- Isometry invariant
- Simple distance computation of the signatures
- No user interaction
New Signature: Shape-DNA
(Reuter, Wolter, Peinecke 2005)

We use the (normed) $n$-dim vector of the smallest $n$ eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of the Laplace operator $\Delta$ as the signature:

- Invariant wrt translation, rotation and (where required) scaling
- No registration necessary
- Surfaces & solids (even with cavities), arbitrary genus
- Independent of representation
- Isometry invariant
- Simple distance computation of the signatures
- No user interaction
New Signature: Shape-DNA
(Reuter, Wolter, Peinecke 2005)

We use the (normed) \( n \)-dim vector of the smallest \( n \) eigenvalues \((\lambda_1, \ldots, \lambda_n)\) of the Laplace operator \( \Delta \) as the signature:

- Invariant wrt translation, rotation and (where required) scaling
- No registration necessary
- Surfaces & solids (even with cavities), arbitrary genus
- Independent of representation
- Isometry invariant
- Simple distance computation of the signatures
- No user interaction
New Signature: Shape-DNA
(Reuter, Wolter, Peinecke 2005)

We use the (normed) $n$-dim vector of the \textbf{smallest $n$ eigenvalues} $(\lambda_1, \ldots, \lambda_n)$ of the Laplace operator $\Delta$ as the signature:

- Invariant wrt translation, rotation and (where required) scaling
- No registration necessary
- Surfaces & solids (even with cavities), arbitrary genus
- Independent of representation
- Isometry invariant
- Simple distance computation of the signatures
- No user interaction
Variational Formulation - FEM

Multiply Helmholtz equation with test functions $\varphi$, then integrate and apply Greens Formula (Variational Form):

$$
\varphi\Delta f = -\lambda\varphi f
\Leftrightarrow \int\int \varphi\Delta f \, d\sigma = -\lambda \int\int \varphi f \, d\sigma
\Leftrightarrow \int\int Df \, G^{-1} \, (D\varphi)^T \, d\sigma = \lambda \int\int \varphi f \, d\sigma
$$

with $Df = (\partial_1 f, \partial_2 f, \ldots)$.

Approximating $f \approx \sum U_i F_i$ (where $F_i$ form functions):

yields: $AU = \lambda BU$

with the matrices (sparse, symmetric, positiv semi-definit):

$$
A = (a_{lm}) := (\int\int (DF_i) \, G^{-1} \, (DF_m)^T \, d\sigma),
B = (b_{lm}) := (\int\int F_i F_m \, d\sigma).
$$

Solve with Lanczos Method from ARPACK
Variational Formulation - FEM

Multiply Helmholtz equation with test functions \( \varphi \), then integrate and apply Greens Formula (Variational Form):

\[
\varphi \Delta f = -\lambda \varphi f
\]

\[
\Leftrightarrow \int \int \varphi \Delta f \, d\sigma = -\lambda \int \int \varphi f \, d\sigma
\]

\[
\Leftrightarrow \int \int Df \, G^{-1} (D\varphi)^T \, d\sigma = \lambda \int \int \varphi f \, d\sigma
\]

with \( Df = (\partial_1 f, \partial_2 f, \ldots) \).

Approximating \( f \approx \sum U_l F_l \) (where \( F_l \) form functions):

yields: \( AU = \lambda BU \)

with the matrices (sparse, symmetric, positiv semi-definit):

\[
A = (a_{lm}) := \left( \int \int (DF_l) \, G^{-1} (DF_m)^T \, d\sigma \right),
\]

\[
B = (b_{lm}) := \left( \int \int F_l F_m \, d\sigma \right).
\]

Solve with Lanczos Method from ARPACK
Example for the exactness

Param. Sphere
5448 DOF (4s)

Facetted Sphere
11522 DOF (5s)

Exact values

\[ \lambda_n = (n - 1)n \]

Multiplicities:
\[ \text{mult} = 2n - 1 \]

0, 2, 2.0001, 2.0001, 6, 6.000008, 6.000008, 6.0005, 6.0005 ... 
\[ \lambda_{100} = 90.034 \ldots \] 
\( (107.02 \text{ linear}) \ldots \)

0, 2.0047, 2.0047, 2.0054, 6.014, 6.014, 6.015, 6.015, 6.016 ... 
90.236 
\( (97.884 \text{ linear}) \ldots \)
Dilation

If $M$ is scaled by $s$, spectrum is scaled by $s^{-2}$ (in any dimension).
Normalization

Two classes of spectra of spheres and ellipsoids with noise
blue: noisy spheres, red: noisy ellipsoids

- Unnormalized: Mainly differences in area/volume
- Normalized: Similarity detection
Normalization

Zoom-ins on the two aligned spectra classes:

blue: noisy spheres, red: noisy ellipsoids

Area/volume normalization shows if additional shape differences exist.
The spectrum depends **continuously** on the shape.
Continuous Dependency on Deformation

MDS Plot 2D - Medial Bar Deformation
Isometry Invariance

Isometric objects have the same spectrum!

- Spectrum is independent of object’s spacial position.
Isometry Invariance

(Modes courtesy of Miralab)
Overview

1. **Shape**
   - Hearing Shape
   - Comparing and Identifying Shape
   - Signatures

2. **Shape-DNA**
   - Laplace-Spectrum as a Signature
   - Implementation
   - Properties of the Spectrum

3. **Applications**
   - Identification and Similarity Detection
   - Global Analysis of Medical Data
Different representations $\Rightarrow$

- challenging to identify a protected object
- challenging to retrieve a specific object from DB
Triangulation of Deformed Spheres

Shape
Hearing Shape
Comparing and Identifying Shape Signatures

Shape-DNA
Laplace-Spectrum as a Signature
Implementation
Properties of the Spectrum

Applications
Identification and Similarity Detection
Global Analysis of Medical Data
Triangulation of Deformed Spheres
Deformed Spheres in 3D

For solid bodies in $\mathbb{R}^3$ isometry is equivalent to congruency.
Global Shape Analysis of Medical Data

Global Shape Analysis on caudate nucleus (Brain MRI)

Populations:

**SPD**
32 female subjects diagnosed with Schizotypal Personality Disorder (SPD)

**NC**
29 female normal control (NC) subjects

(Harvard Medical - Psychiatry NeurolImaging Laboratory)
Rendering of the Caudate Nucleus

Coronal view.

Involved in memory function, emotion processing, and learning.

The caudate nucleus was delineated manually by an expert.
Shape comparison either on volumetric data (e.g. tetrahedrization or directly on binary voxel data):

or extraction of (smoothed) iso surfaces:
Permutation tests to compare two populations (200,000 permutations)

- Unnormalized shapes show statistically significant differences (expected: volume, area differences).
- Stat. sign. differences with normalized shapeDNA indicate true shape differences.
- For 3D voxels Neumann spectra indicate differences in smaller features.
Permutation tests to compare two populations (200,000 permutations)

- Unnormalized shapes show statistically significant differences (expected: volume, area differences).
- Stat. sign. differences with normalized shapeDNA indicate true shape differences.
- For 3D voxels Neumann spectra indicate differences in smaller features.
Permutation tests to compare two populations (200,000 permutations)

- Unnormalized shapes show statistically significant differences (expected: volume, area differences).
- Stat. sign. differences with normalized shapeDNA indicate true shape differences.
- For 3D voxels Neumann spectra indicate differences in smaller features.
Permutation tests to compare two populations (200,000 permutations)

- Unnormalized shapes show statistically significant differences (expected: volume, area differences).
- Stat. sign. differences with normalized shapeDNA indicate true shape differences.
- For 3D voxels Neumann spectra indicate differences in smaller features.
Images (Peinecke, Wolter, Reuter 2007)

Height function:

\[ \Delta f = -\lambda \rho f \] with the mass-density function \( \rho \).
Images (Peinecke, Wolter, Reuter 2007)

Height function:

\[ \Delta f = -\lambda \rho f \] with the mass-density function \( \rho \).
Vectorfields

- We have seen examples in scalar fields (MRI data, images)
- Extension to vector fields $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $m > 1$?
- $f$ generally not a parametrization of a manifold
- extending the $m$ coordinates of the function $f$ with the $n$ parameter values:

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f_1, \ldots, f_m)$$

yields a parametrization of a manifold.
Vectorfields

- We have seen examples in scalar fields (MRI data, images)
- Extension to vector fields $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $m > 1$?
- $f$ generally not a parametrization of a manifold
- extending the $m$ coordinates of the function $f$ with the $n$ parameter values:

$$F(x_1, ..., x_n) = (x_1, ..., x_n, f_1, ..., f_m)$$

yields a parametrization of a manifold.
Vectorfields

- We have seen examples in scalar fields (MRI data, images)
- Extension to vector fields \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) with \( m > 1 \)?
- \( f \) generally not a parametrization of a manifold
- extending the \( m \) coordinates of the function \( f \) with the \( n \) parameter values:

\[
F(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f_1, \ldots, f_m)
\]

yields a parametrization of a manifold.
Vectorfields

- We have seen examples in scalar fields (MRI data, images)
- Extension to vector fields $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $m > 1$?
- $f$ generally not a parametrization of a manifold
- extending the $m$ coordinates of the function $f$ with the $n$ parameter values:

$$F(x_1, ..., x_n) = (x_1, ..., x_n, f_1, ..., f_m)$$

yields a parametrization of a manifold.
Conclusion

- ShapeDNA has many desired properties for shape matching
  - Mainly: isometry invariance
  - Can be computed very accurately with FEM
  - Volumetric spectra are feasible for 3D shape analysis
  - Method universally applicable for imaging and CAD applications
  - Comparison of shape based on feature size (frequency of eigenfunctions)
**Conclusion**

- **ShapeDNA** has many desired properties for shape matching
- **Mainly:** isometry invariance
- Can be computed very accurately with FEM
- Volumetric spectra are feasible for 3D shape analysis
- Method universally applicable for imaging and CAD applications
- Comparison of shape based on feature size (frequency of eigenfunctions)
Conclusion

- ShapeDNA has many desired properties for shape matching
- Mainly: isometry invariance
- Can be computed very accurately with FEM
- Volumetric spectra are feasible for 3D shape analysis
- Method universally applicable for imaging and CAD applications
- Comparison of shape based on feature size (frequency of eigenfunctions)
Conclusion

- ShapeDNA has many desired properties for shape matching
- Mainly: isometry invariance
- Can be computed very accurately with FEM
- Volumetric spectra are feasible for 3D shape analysis
- Method universally applicable for imaging and CAD applications
- Comparison of shape based on feature size (frequency of eigenfunctions)
Conclusion

- ShapeDNA has many desired properties for shape matching
- Mainly: isometry invariance
- Can be computed very accurately with FEM
- Volumetric spectra are feasible for 3D shape analysis
- Method universally applicable for imaging and CAD applications
- Comparison of shape based on feature size (frequency of eigenfunctions)
Conclusion

- ShapeDNA has many desired properties for shape matching
  - Mainly: isometry invariance
  - Can be computed very accurately with FEM
  - Volumetric spectra are feasible for 3D shape analysis
  - Method universally applicable for imaging and CAD applications
  - Comparison of shape based on feature size (frequency of eigenfunctions)
Thanks

Thank you very much for your attention!

Publications can be found at http://reuter.mit.edu