Interactions Between Quantum Graphs and Harmonic Analysis

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Abstract
Quantum graphs arise as models for thin physical networks. Their Laplace operators blend a one dimensional local structure with a graphical global structure through junction coupling conditions. When edge lengths are equal, or rationally related, the eigenvalues and trigonometric eigenfunctions have a surprisingly simple structure. As a consequence, they exhibit aspects of function theory usually associated with the circle, including a continuous graph Fast Fourier Transform.
Two pictures
Mention less regular networks -
road networks
river systems
human arterial tree
1 Differential equations on graphs

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{wire_screen.png}
\caption{Wire screen}
\end{figure}

Use the 1-d heat equation for each wire,

\[ \frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x), \quad u(0, x) = f(x). \]

Couple edges \(e_j\) and \(e_k\) meeting at \(v\),

\[ f_j(v) = f_k(v), \quad \sum f'_j(v) = 0, \]

the sum over all edges meeting at \(v\), and derivatives computed with outward pointing local coordinates.
2 Quantum graph framework

Finite graph, finite length edges. Edges are identified with intervals $e = [a_e, b_e]$. Hilbert space $\bigoplus_j L^2(e_j)$ with inner product

$$\langle f(x), g(x) \rangle = \frac{1}{\sum e (b_e - a_e)} \sum_e \int_{a_e}^{b_e} f_e(x) \overline{g_e(x)} \, dx.$$ 

Laplace operator $-\partial^2 / \partial x^2$ with above domain is a nonnegative self adjoint operator with compact resolvent. Eigenfunctions are trigonometric on each edge.

The 'Sturm-Liouville' or 'Fourier series' theory extends (beyond $L^2$) (M. Baker and R. Rumely)

Literature disclaimer

By inserting 'invisible' vertices and rescaling, make rational edge length graphs have edge lengths 1.
3 Discrete Graph Operators

With vertices $v_1, \ldots, v_N$ the adjacency matrix $A$ is

$$A_{jk} = \begin{cases} 1, & \{v_j, v_k\} \in \mathcal{E}, \\ 0, & \{v_j, v_k\} \notin \mathcal{E} \end{cases}.$$ 

There is also an inner product

$$\langle f, g \rangle = \sum_{v \in \mathcal{V}} \deg(v)f(v)\overline{g(v)},$$

$\deg(v) = \text{number of incident edges}.$

A self adjoint discrete Laplacian with this inner product is

$$\Delta_1 = I - D^{-1}A, \quad Df(v) = \deg(v)f(v).$$
4 Edge lengths 1 - Remarkable facts

(von Below, Cattaneo, Friedman-Tillich)

The eigenspace $E(\lambda)$ for $-\partial^2/\partial x^2$ has 'periodicity':

**Proposition 4.1.** If $\omega = \sqrt{\lambda} > 0$,

$$\dim E(\omega^2) = \dim E([\omega + 2m\pi]^2), \quad m = 1, 2, 3, \ldots.$$ 

Using 'vertex evaluation map' we find

**Theorem 4.2.** If $\lambda \notin \{n^2\pi^2 \mid n = 0, 1, 2, \ldots\}$, then $\lambda$ is an eigenvalue of $-\partial^2/\partial x^2$ if and only if $\mu = 1 - \cos(\sqrt{\lambda})$ is an eigenvalue of $\Delta_1$, with the same multiplicity.

The cases $\lambda \in \{n^2\pi^2 \mid n = 1, 2, \ldots\}$ are also 'combinatorial'. The edge space has the cycle subspace $Z_0(\mathcal{G})$. Let $Z_1$ be the set of functions $f : \mathcal{E} \to \mathbb{C}$ with

$$\sum_{e \approx v} f(e) = 0, \quad v \in \mathcal{V}.$$ 

Let $E_0(n^2\pi^2) \subset E(n^2\pi^2)$ be the eigenfunctions of $\partial^2/\partial x^2$ vanishing at the vertices.

**Theorem 4.3.**

$$\dim(Z_0(\mathcal{G})) = \dim E_0((2n\pi)^2).$$

$$\dim(Z_1) = \dim E_0((2n - 1)\pi)^2).$$
5 Graph refinements

Thinking like a numerical analyst, let's sample the length 1 graph edges uniformly.

![Refinement of a graph](image)

Figure 3.1: Refinement of a graph

The operator $\Delta_\infty = -\partial^2/\partial x^2$ stays fixed since the new vertex conditions do not change the domain.

The graphs have changed, giving new adjacency matrices and degree operators.

By the v.B.C.,F.-T. theorem orthogonal eigenspaces of $-\partial^2/\partial x^2$ descend by sampling to orthogonal eigenspaces of the 'new $\Delta_1$' if the corresponding eigenvalues are distinct.

This generalizes the Fourier series - DFT coupling of $S^1$. 
6 Discrete Graph Fourier analysis

After sampling edge $e \in G_1$ becomes $N$ edges in $G_N$ with vertex space $H_N$, discrete Laplacian

$$
\Delta_N = N^2(I - D^{-1}A), \quad \Delta_N : H_N \rightarrow H_N,
$$

and inner product

$$
\langle f, g \rangle_N = \frac{1}{2NN_E} \sum_v \text{deg}(v)f(v)\overline{g(v)}.
$$

Eigenspaces are $E_N(\lambda)$, resp. $E_\infty(\lambda)$. Let $E_p(n^2\pi^2)$ be the subspace having the form $C\cos(n\pi x)$ on each edge. Let $S_N \subset L^2(G_\infty)$ denote the subspace

$$
S_N = \text{span}\{E_p(N^2\pi^2), E_\infty(\lambda), 0 \leq \lambda < N^2\pi^2\}.
$$

Proposition 6.1. The restriction $R_N : S_N \rightarrow H_N$ is a bijection. For $0 \leq \lambda < N^2\pi^2$ this map takes distinct orthogonal eigenspaces $E_\infty(\lambda)$ of $\Delta_\infty$ onto distinct orthogonal eigenspaces $E_N(N^2(1 - \cos(\sqrt{\lambda}/N)))$ of $\Delta_N$, and $R_N$ takes $E_p(N^2\pi^2)$ onto $E_N(2N^2)$. 

8
We'd like an FFT using sampled $\Delta_\infty$ eigenfunctions. Good bases generated by 'frequency increase'. Multiplicity of eigenvalues complicates image orthogonality within eigenspaces.

**Theorem 6.2.** There is a Fourier transform

$$\mathcal{F}_N : \mathbb{H}_N \rightarrow \mathbb{C}^M, \quad M = \dim(\mathbb{H}_N)$$

satisfying

$$\mathcal{F}_N(\Delta_N f) = \{\mu_{m,k}\mathcal{F}(f)_{m,k}\},$$

isometric on $\mathbb{C}^M$ with a modified inner product.

If $N$ is a power of 2, then $\mathcal{F}_N(f)$ and its inverse can be computed in time $O(N \log_2(N))$.

Question: Are leafless equilateral graph eigenspaces $E(\lambda)$ for $\lambda \neq n^2\pi^2$ 'generically' simple? Friedlander: arbitrary positive real lengths.
7 A family of examples

Basic examples - complete bipartite graphs $K(m, 2)$ on $m, 2$ vertices.

The graph $K_{4,2}$

Local model for any graph.
Obtain from the polar coordinate 2-sphere by uniform angular sampling.
Rotationally symmetric eigenfunctions $\cos(k\pi x)$.
Rest $\sin(k\pi x) \exp(2\pi i \frac{im}{M})$. 

8 Rework $S^2$ trapezoidal rule analysis

'Continuum limit' of $K(m, 2)$ Laplacians leads to polar $S^2$ Laplacian $\Delta_p = \partial^2/\partial x^2 + \partial^2/\partial y^2$ with nonlocal polar condition

$$\int_0^1 \partial_x f(0, y) \, dy = 0 = \int_0^1 \partial_x f(1, y) \, dy.$$  

Trapezoidal rule

$$T(M, N, f) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n} f(x_m, y_n) \sim \int_{I^2} f(x, y)$$

exact on eigenfunctions with corresponding index range.

Sobolev spaces $H^s$ from domain of $\Delta_p^{s/2}$ provide rates,

**Theorem 8.1.** If $f(x, y) \in H^s$ and $L = \min(M, N)$, then

$$\int_{I^2} f(x, y) - T(M, N, f) = O(L^{2-s}).$$

Novelty for singular function estimates. The singular function $\cos(2\pi y)$ on $S^2$, which is not continuous at the poles, converts to the polar integrand $f(x, y) = 2\pi^2 \sin(\pi x) \cos(2\pi y)$, which is in all $H^s$. 

11
References


