Spectral Graph Wavelets on the Cortical Connectome and Regularization of the EEG Inverse Problem

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International Conference on Industrial and Applied Mathematics
11:30-11:55 July 22, 2011
Overview

Spectral Graph Wavelet Transform (SGWT)

Motivations from Classical Wavelet Analysis

Spectral Graph Theory

SGWT construction

EEG source estimation

Electrical head modeling

Cortical connectome graph construction

Sparse representation with Cortical Spectral Graph Wavelets
SGWT : wavelets on weighted graphs

Weighted graphs:

- $N$ vertices
- Symmetric adjacency matrix

$$A_{i,j} = w_{i,j} \quad (w_{i,j} \geq 0)$$

Wavelets on weighted graph:

- Linear, multiscale representation for functions on vertices $f \in \mathbb{R}^N$

$$\psi_\gamma \in \mathbb{R}^N$$ localized in space and frequency

$$c_\gamma = \langle \psi_\gamma, f \rangle$$

Why?

- Wavelets: very useful, but classically limited to Euclidean space

- Graphs: flexibly model complicated data domains
Preview: some SGWT graph wavelets
Classical wavelet analysis

Signal $f \in L^2(\mathbb{R})$

“mother” wavelet $\psi$

Analysis : wavelet coefficients

$$W_f(t, u) = \langle \psi_{t, u}, f \rangle$$

Reconstruction (Orthogonal Wavelet case)

$$f = \sum_{n,j} W_f(2^j, 2^j n) \psi_{2^j, 2^j n}(x)$$

Signal manipulation often easier in wavelet transform space
Towards Wavelets on Graphs

Problem: dilation and translation on irregular Graph?

Wavelet transform in Fourier domain:

\[ W_f(t, x) = f \ast \overline{\psi}_t \]
\[ W_f(t, w) = \hat{f}(\omega) \hat{\psi}^*(t\omega) \]

Individual wavelets: apply operator to \( \delta_u(x) = \delta(x - u) \)

\[ \psi_{t,u}(-x) = W_{\delta_u}(t, x) \]
\[ = \mathcal{F}^{-1} \begin{pmatrix} \hat{\delta}_u(\omega) & \hat{\psi}^*(t\omega) \end{pmatrix} \]

Dilate operator \( \hat{\psi}^*(t\omega) \) in frequency domain, then localize

Analog of Fourier transform on weighted graphs => Spectral graph theory
Spectral Graph Theory

Degree matrix
\[ D_{i,i} = \sum_{k=1}^{N} A_{k,i} \]

Graph Laplacian
\[ L = D - A \]

Spectral decomposition of \( L \)
\[ L \chi_l = \lambda_l \chi_l \quad 0 = \lambda_1 \leq \lambda_1 \leq \ldots \leq \lambda_N \]
\( \{\chi_l\}_{l=1}^{N} \) orthonormal basis

Graph “Fourier transform”
\[ \hat{f}(l) = \langle \chi_l, f \rangle = \sum_n \chi_l^*(n) f(n) \]

Graph “inverse Fourier transform”
\[ f = \sum_l \hat{f}(l) \chi_l \]
Spectral Graph Wavelets

Wavelet kernel \( g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \)

analogous to \( \hat{\psi}^*(\omega) \)

Wavelet operator (at scale \( t_j \))

\[
T_g^{t_j} = g(t_j L) : \mathbb{R}^N \rightarrow \mathbb{R}^N
\]

Action defined on eigenvectors

\[
T_g^{t_j} \chi_l = g(t_j \lambda_l) \chi_l
\]

Wavelets

\[
\psi_{j,n} = T_g^{t_j} \delta_n
\]

SGWT coefficients at scale \( j \)

\[
T_g^{t_j} f = \sum_l g(t_j \lambda_l) \hat{f}(l) \chi_l
\]
Spectral Graph Wavelets

Scaling function band

\[ T_h = h(L) \]
\[ \phi_n = T_h \delta_n \]

K-fold Overcomplete

\[ W : \mathbb{R}^N \rightarrow \mathbb{R}^{KN} \]

Fast SGWT

Polynomial approximation of \( g(t_j x) \)

Avoids diagonalization of \( L \)
Application to EEG Source Estimation
Electroencephalography (EEG)

EEG signal: arises from dipole current sources + volume conduction

Source estimation: Infer current sources $J$ from electrode measurements $\Phi$

Access to subject specific brain anatomy:

MRI: measure tissue geometry

Diffusion tensor imaging (DTI) + tract tracing: measure brain connectivity

How to exploit connectivity knowledge? Sparsity in cortical graph wavelet basis.
EEG forward problem

Quasi-static Maxwell PDE

\[ \nabla \cdot (\sigma \nabla \phi) = s \]

\[ (\sigma \nabla \phi) \cdot \vec{n} = 0 \text{ on boundary} \]

\( \phi \) : potential

\( s \) : current sources

\( \sigma(x, y, z) \) : tissue conductivity

Finite-difference solver, on 1mm 3D grid

Lead field matrix \( K \)

\( K_{ij} \): solution at \( e_i \) with unit source at \( d_j \)

\[ \Phi = KJ \]

\( J \in \mathbb{R}^{Nd} \): source current amplitudes

\( \Phi \in \mathbb{R}^{Ne} \): electrode voltages
Cortical connectome graph

Cortical grey matter (neuron cell bodies) connected by white matter fibers (axons)

Diffusion along fiber directions

Diffusion Tensor Imaging (DTI) measures water diffusion direction at every voxel

Tractography

dominant eigenvector $v_1$ reveals fiber orientation
compute streamlines, from seed points

( a little more complicated, in practice ... )

using method developed by S Warfield, B Scherrer, Children’s Hospital, Harvard
Cortical connectome graph

seed in white matter, retain tracts connecting cortex to cortex

global (tract-based) connectome $A^t$  

$$a^t_{i,j} = \sum_{k \text{ connecting } i \text{ and } j} \frac{1}{\text{length}(k)}$$

local (adjacency-based) connectome $A^l$  

$$a^l_{i,j} = \text{length of boundary between patches } i, j$$

hybrid connectome $A$  

$$A = \lambda_t A^t + \lambda_l A^l$$

showing 12,124 out of 775,939 cortical-cortical tracts
Cortical connectome graph

- Seed in white matter, retain tracts connecting cortex to cortex

- Global (tract-based) connectome $A^{tr}$
  \[
  a_{i,j}^{tr} = \sum_{k \text{ connecting } i \text{ and } j} \frac{1}{\text{length}(k)}
  \]

- Local (adjacency-based) connectome $A^{loc}$
  \[
  a_{i,j}^{loc} = \text{length of boundary between patches } i, j
  \]

- Hybrid connectome $A$
  \[
  A = \lambda_{tr} A^{tr} + \lambda_{loc} A^{loc}
  \]

Showing 12,124 out of 775,939 cortical-cortical tracts
Cortical connectome graph

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$$a^{loc}_{i,j} = \text{length of boundary between patches } i, j$$

hybrid connectome $A$

$$A = \lambda_{tr} A^{tr} + \lambda_{loc} A^{loc}$$

showing 12,124 out of 775,939 cortical-cortical tracts
cortex patches, showing patch centers
Global (tract-based) connectome graph
Local connectome graph
EEG inverse problem

linear superposition:

\[ \Phi = KJ \]

Inverse problem: Given \( \Phi \), find \( J \)

\( N_e \) equations, \( N_d \) unknowns \( \Rightarrow \) infinitely many possibilities for \( J \)!

Find \( J \) minimizing

\[ ||\Phi - KJ||^2 + f(J) \]

\( f(J) \):
small for “good” \( J \)
large for “bad” \( J \)

data fidelity prior

how to build prior using connectivity?
Sparse representation with Cortical Graph Wavelets

Construct SGWT from hybrid connectome $A$

\[ W : \mathbb{R}^{N_d} \rightarrow \mathbb{R}^{KN_d} \]

\[ J = \sum_i c_i \psi_i = W^T c \]

prior: $\ell_1$ penalty on $c$

\[ c^* = \arg\min_c \|\Phi - KW^T c\|_2^2 + \lambda\|c\|_1 \]

\[ J^* = W^T c^* \]

(P) is convex, L1-LS program

solve with truncated Newton interior point method

Kim, Koh, Lustig, Boyd, Gorinevsky 2007

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Motor Potential study

Validation: investigate source estimates for paradigm where we “know” where the sources should be

Experimental paradigm:

Press button (RT, LT, RP, LP)

EEG recording setup:

256 - channel (257 electrodes) sensor net, 250 Hz average locked to button press onset (ERP)

expect activation in contralateral motor cortex
Preliminary Results

40 ms before left thumb press

subject 1470

minimum norm (l2 penalty)

proposed (wavelet l1 penalty)

dipole l1 penalty
subject
1490

76 ms before right thumb press

minimum norm
(l2 penalty)

proposed
(wavelet l1 penalty)

dipole l1 penalty
Future Work

Explore parameters (connectome graph construction, SGWT ...)

Sensible automatic selection of regularization parameters (L-curve)

Spatiotemporal estimation via spatiotemporal graph

More appropriate convex solver (path following for $\lambda$)?

Alternative convex program formulations

More subjects / different experimental paradigms
Acknowledgements

Allen Malony               NIC, University of Oregon
Don Tucker                  EGI / NIC, University of Oregon
Pierre Vandergheynst    EPFL, Switzerland
Rémi Gribonvale           INRIA, France
Adnan Salman               NIC, University of Oregon
Kai Li                          NIC, University of Oregon
Simon Warfield             CRL, Children’s Hospital, Harvard
Benoit Scherrer         CRL, Children’s Hospital, Harvard
FIN
Chebyshev polynomials for fast SGWT

\[ g(t_j x) \approx p_j(x) = \sum_{k=0}^{M} c_{j,k} T_k(x) \]

Chebyshev approx for M=10

\[ g(t_j L)f \approx p_j(L)f = \sum_{k=0}^{M} c_{j,k}(T_k(L)f) \]

\[ T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x) \]

\[ T_k(L)f = 2L(T_{k-1}(L)f) - T_{k-2}(L)f \]

computing \( T_k(L)f \) is fast if \( L \) is sparse
Some Cortical Graph Wavelets