Spectral Graph Wavelets on the Cortical Connectome and Regularization of the EEG Inverse Problem

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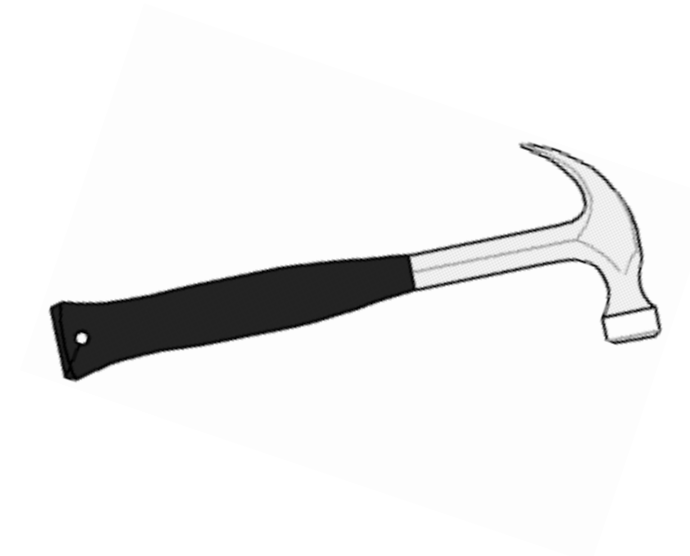
International Conference on
Industrial and Applied Mathematics
11:30-11:55 July 22, 2011

Overview



Spectral Graph Wavelet Transform (SGWT)

Motivations from Classical Wavelet Analysis

Spectral Graph Theory

SGWT construction

EEG source estimation

Electrical head modeling

Cortical connectome graph construction

Sparse representation with Cortical Spectral Graph Wavelets

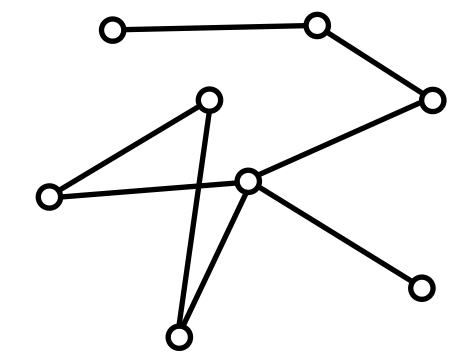
SGWT : wavelets on weighted graphs

Weighted graphs :

N vertices

Symmetric adjacency matrix

$$A_{i,j} = w_{i,j} \quad (w_{i,j} \geq 0)$$



$f(i)$: value on i^{th} vertex

Wavelets on weighted graph:

Linear, multiscale representation for functions on vertices $f \in \mathbb{R}^N$

$\psi_\gamma \in \mathbb{R}^N$ localized in space and frequency

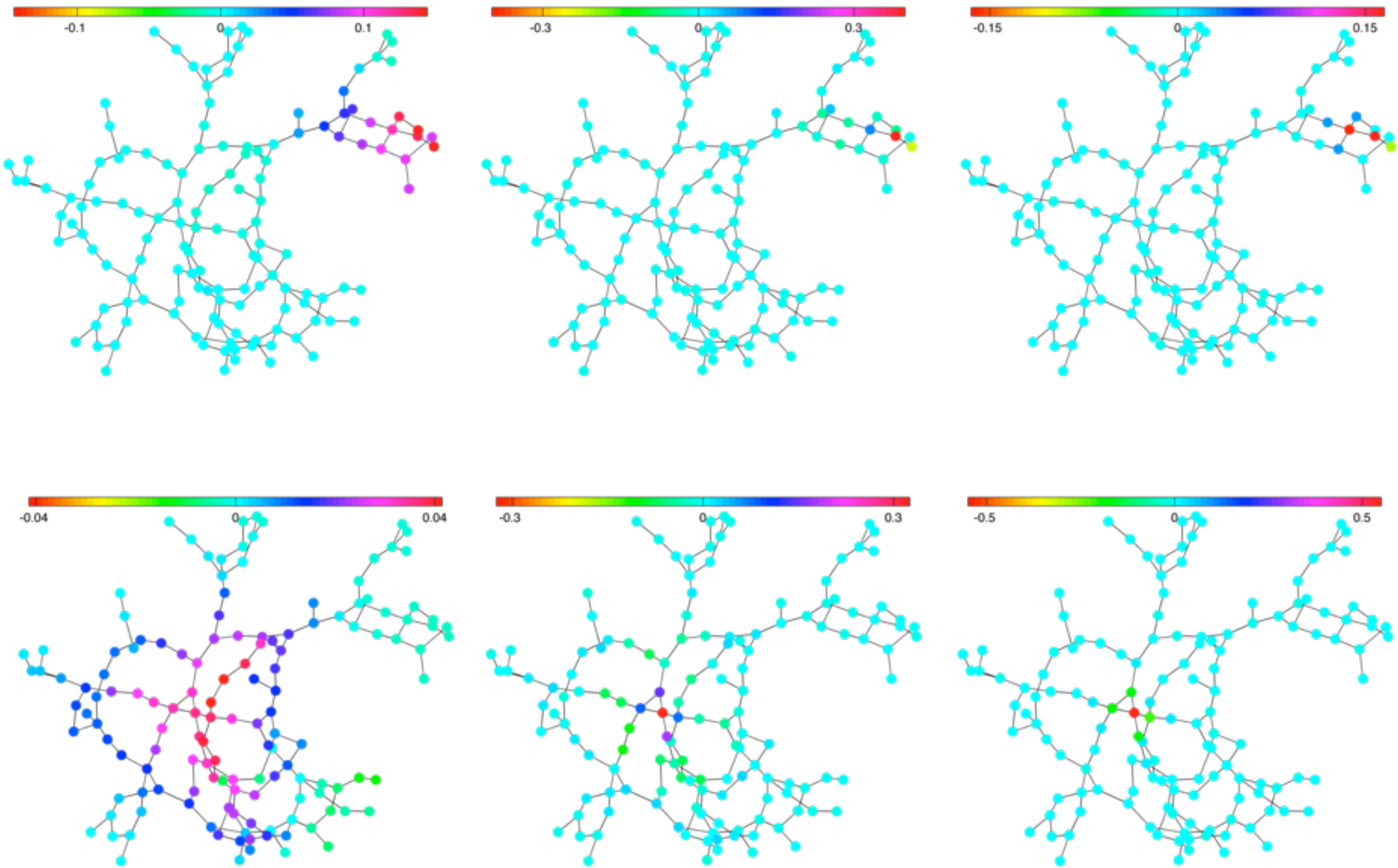
$c_\gamma = \langle \psi_\gamma, f \rangle$

Why?

Wavelets : very useful, but classically limited to Euclidean space

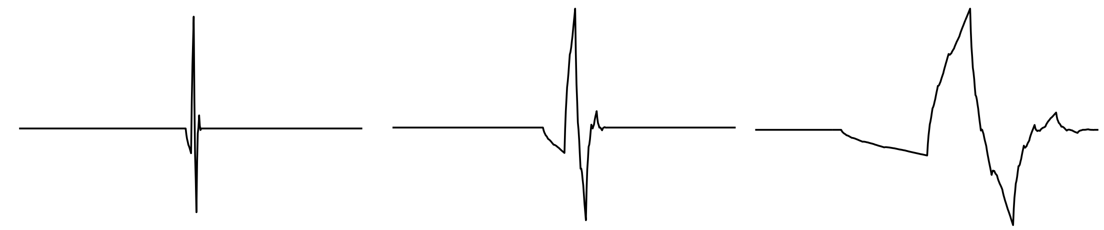
Graphs : flexibly model complicated data domains

Preview : some SGWT graph wavelets



Classical wavelet analysis

Signal $f \in L^2(\mathbb{R})$



"mother" wavelet ψ

translation & dilation

$$\psi_{t,u}(x) = \frac{1}{\sqrt{t}}\psi\left(\frac{x-u}{t}\right)$$

Analysis : wavelet coefficients

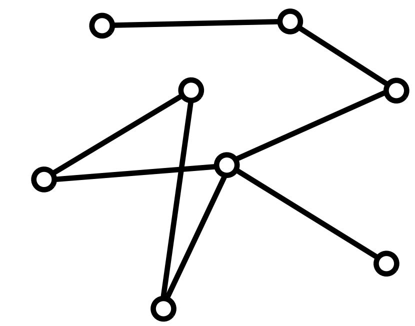
$$W_f(t,u) = \langle \psi_{t,u}, f \rangle$$

Reconstruction (Orthogonal Wavelet case)

$$f = \sum_{n,j} W_f(2^j, 2^j n)\psi_{2^j,2^j n}(x)$$

Signal manipulation often easier in wavelet transform space

Towards Wavelets on Graphs



Problem : dilation and translation on irregular Graph?

Wavelet transform in Fourier domain :

$$W_f(t,x) = f \star \bar{\psi}_t \qquad \bar{\psi}_t(x) = \frac{1}{t}\psi(-\frac{x}{t})$$

$$\widehat{W_f(t,w)} = \hat{f}(\omega)\underline{\hat{\psi}^*(t\omega)}$$

Individual wavelets : apply operator to $\delta_u(x) = \delta(x-u)$

$$\begin{aligned}\psi_{t,u}(-x) &= W_{\delta_u}(t,x) \\ &= \mathcal{F}^{-1}\left(\underbrace{\boxed{\hat{\delta}_u(\omega)}}_{\text{localization}} \; \underbrace{\boxed{\hat{\psi}^*(t\omega)}}_{\text{dilation}} \right)\end{aligned}$$

Dilate operator $\hat{\psi}^*(t\omega)$ in frequency domain, then localize

Analog of Fourier transform on weighted graphs => Spectral graph theory

Spectral Graph Theory

Degree matrix

$$D_{i,i} = \sum_{k=1}^{N} A_{k,i}$$

Graph Laplacian

$$L = D - A$$

Spectral decomposition of L

$$L\chi_l = \lambda_l \chi_l \qquad 0 = \lambda_1 \leq \lambda_1 \leq ... \leq \lambda_N$$

$\{\chi_l\}_{l=1}^{N}$ orthonormal basis

Graph "Fourier transform"

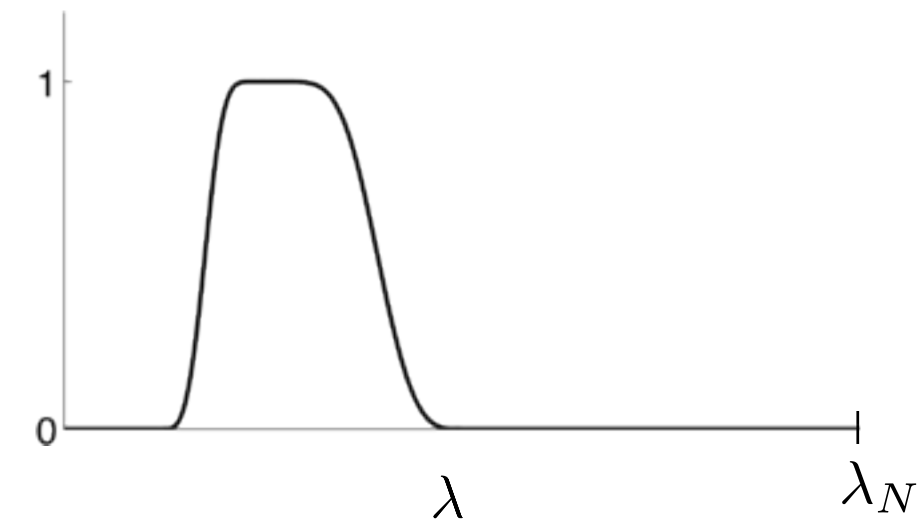
$$\hat{f}(l) = \langle \chi_l, f \rangle = \sum_n \chi_l^*(n) f(n)$$

Graph "inverse Fourier transform"

$$f = \sum_l \hat{f}(l) \chi_l$$

Spectral Graph Wavelets

H., P. Vandergheynst, R. Gribonval 2011
Applied and Computational Harmonic Analysis



Wavelet kernel $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

analogous to $\hat{\psi}^*(\omega)$

Wavelet operator (at scale t_j)

$$T_g^{t_j} = g(t_j L) : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

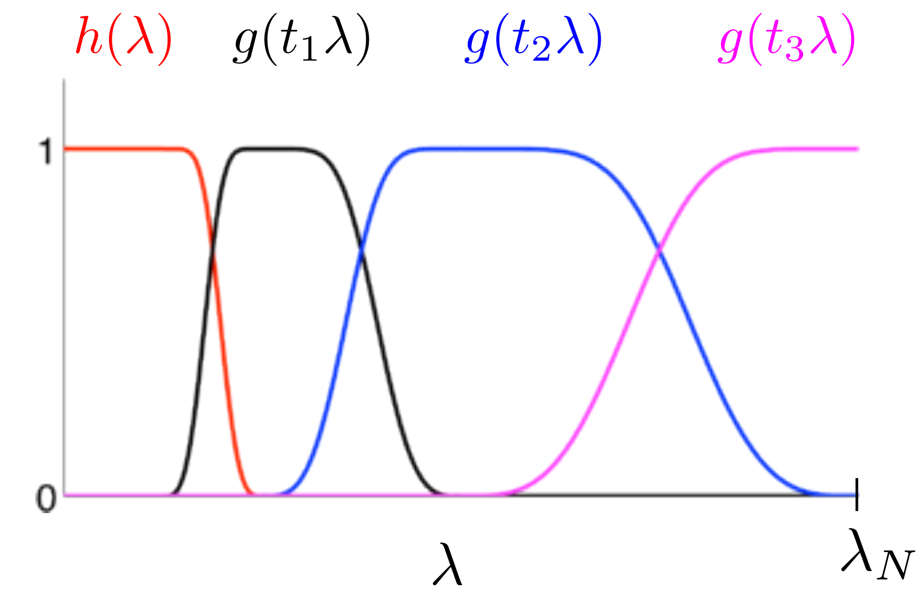
Action defined on eigenvectors

$$T_g^{t_j} \chi_l = g(t_j \lambda_l) \chi_l$$

Wavelets $\psi_{j,n} = T_g^{t_j} \delta_n$

SGWT coefficients at scale j $T_g^{t_j} f = \sum_l g(t_j \lambda_l) \hat{f}(l) \chi_l$

Spectral Graph Wavelets



Scaling function band

$$T_h = h(L)$$

$$\phi_n = T_h \delta_n$$

K-fold Overcomplete

K=4

$$W : \mathbb{R}^N \to \mathbb{R}^{KN}$$

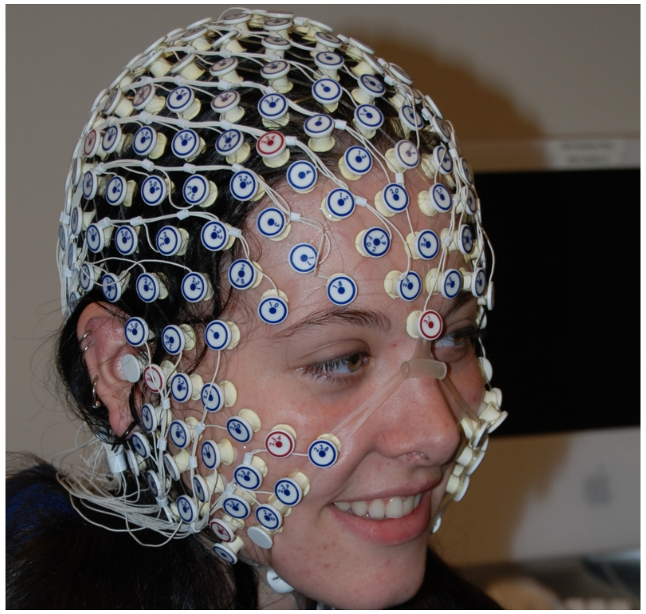
Fast SGWT

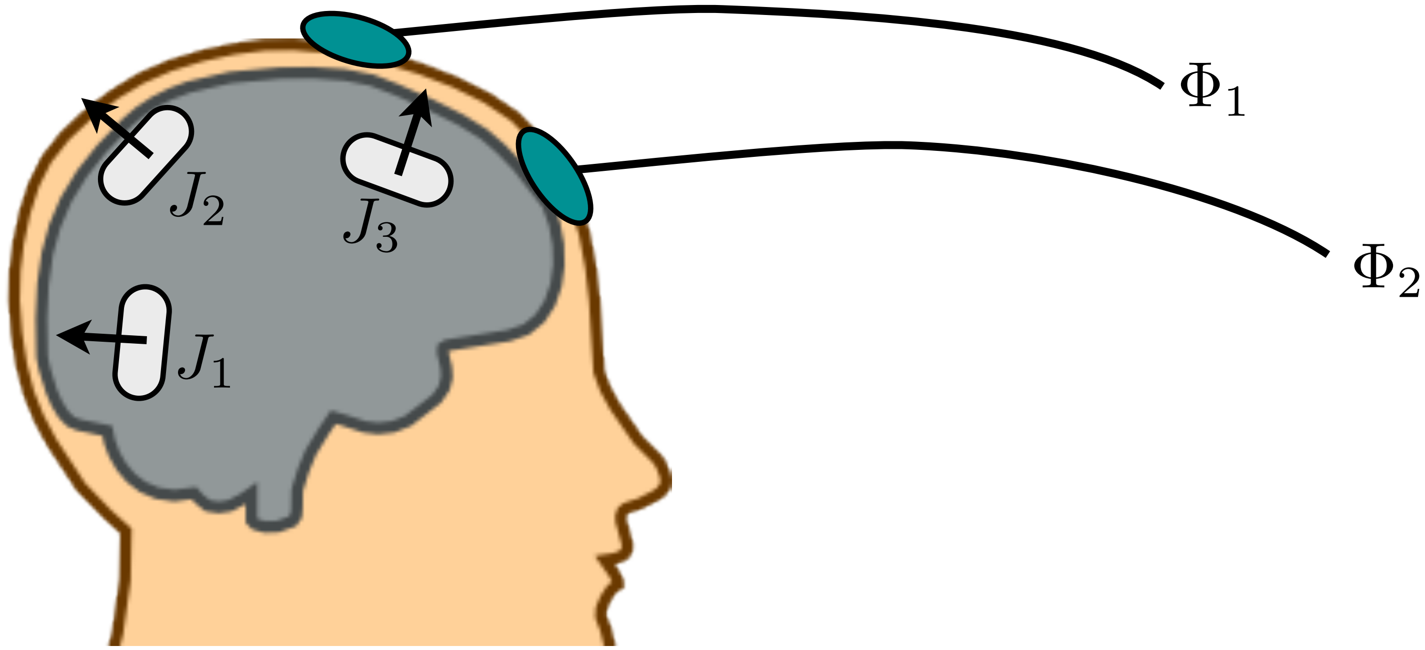
Polynomial approximation of $g(t_j x)$

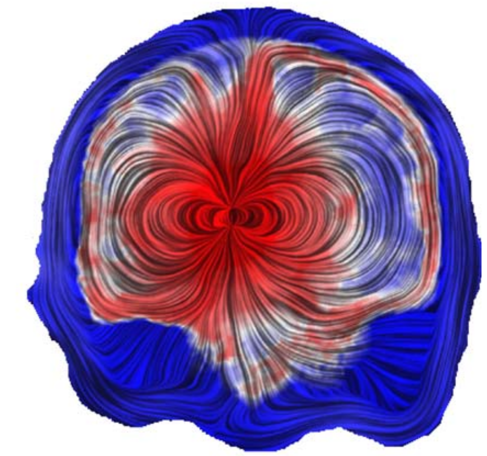
Avoids diagonalization of L

Application to EEG Source Estimation

Electroencephalography (EEG)







EEG signal : arises from dipole current sources + volume conduction

Source estimation : Infer current sources J from electrode measurements Φ

Access to subject specific brain anatomy :

MRI : measure tissue geometry

Diffusion tensor imaging (DTI) + tract tracing : measure brain connectivity

How to exploit connectivity knowledge? Sparsity in cortical graph wavelet basis.

EEG forward problem

Quasi-static Maxwell PDE

$$\nabla \cdot (\sigma \nabla \phi) = s$$

ϕ : potential

s : current sources

$\sigma(x, y, z)$: tissue conductivity

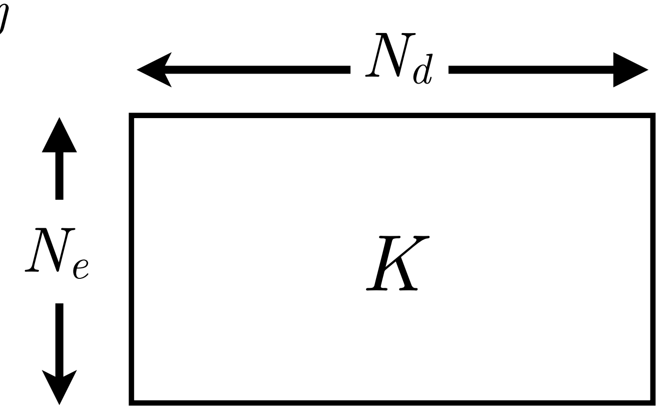
$$(\sigma \nabla \phi) \cdot \vec{n} = 0 \qquad \text{on boundary}$$

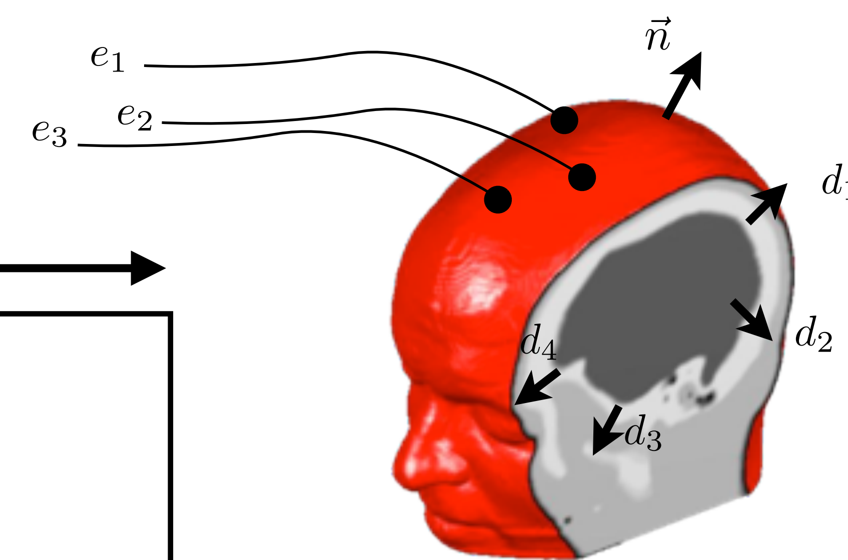
Finite-difference solver, on 1mm 3D grid

Lead field matrix K

K_{ij} : solution at e_i with unit source at d_j

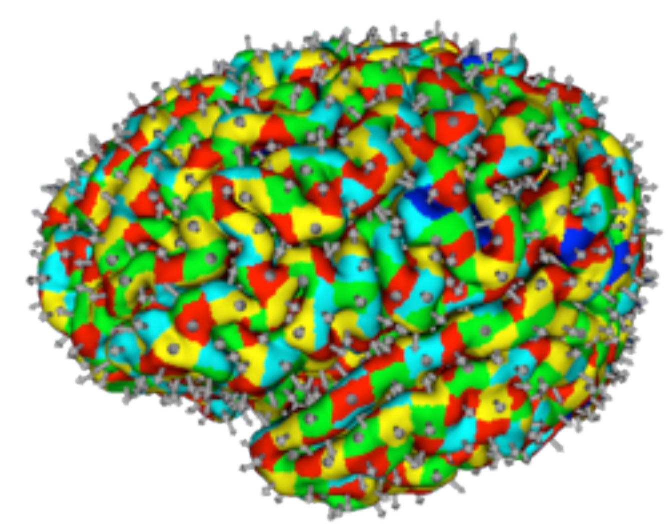
$$\Phi = KJ$$





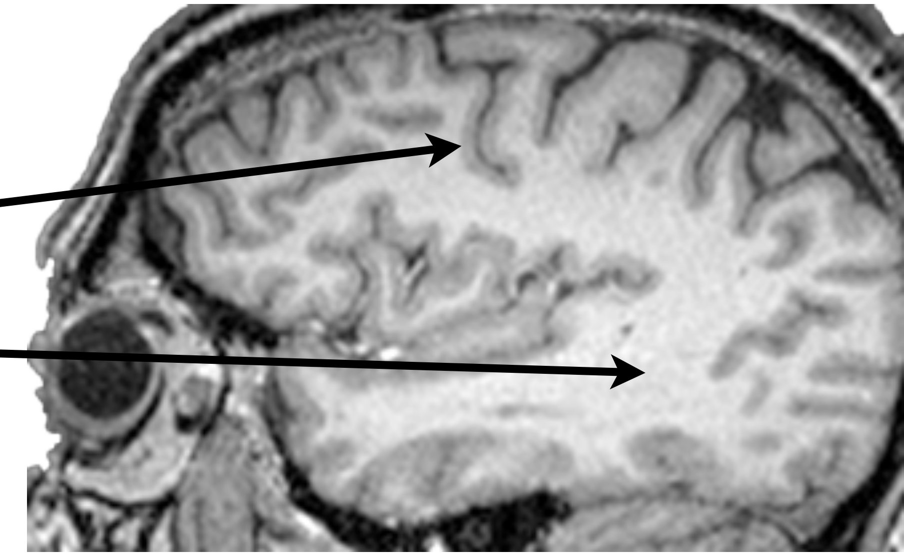
$J \in \mathbb{R}^{N_d}$: source current amplitudes

$\Phi \in \mathbb{R}^{N_e}$: electrode voltages

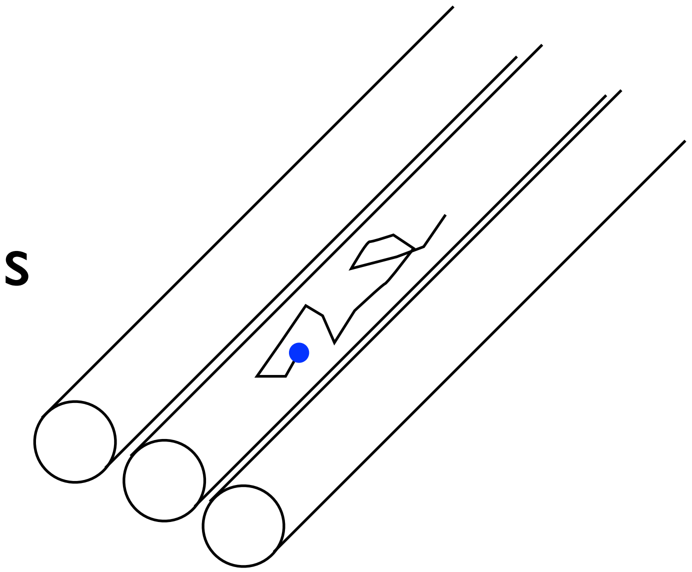


Cortical connectome graph

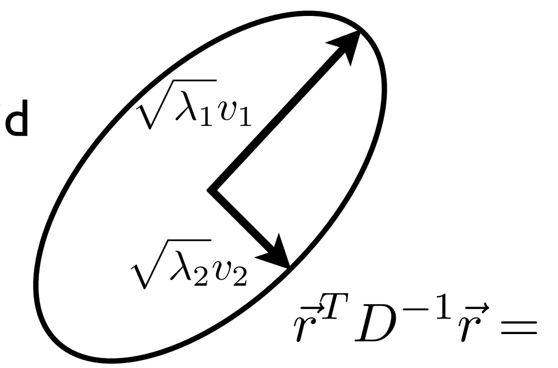
Cortical grey matter (neuron cell bodies)
connected by white matter fibers (axons)



Diffusion along fiber directions



diffusion ellipsoid



Diffusion Tensor Imaging (DTI) measures water diffusion direction at every voxel

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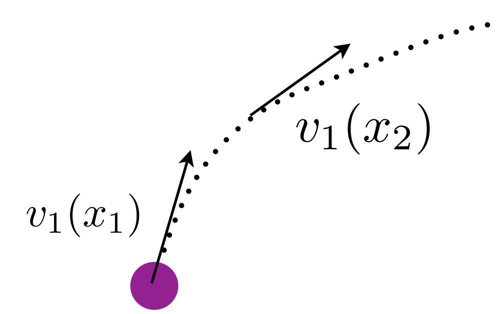
Tractography

dominant eigenvector v_1 reveals fiber orientation

compute streamlines, from seed points

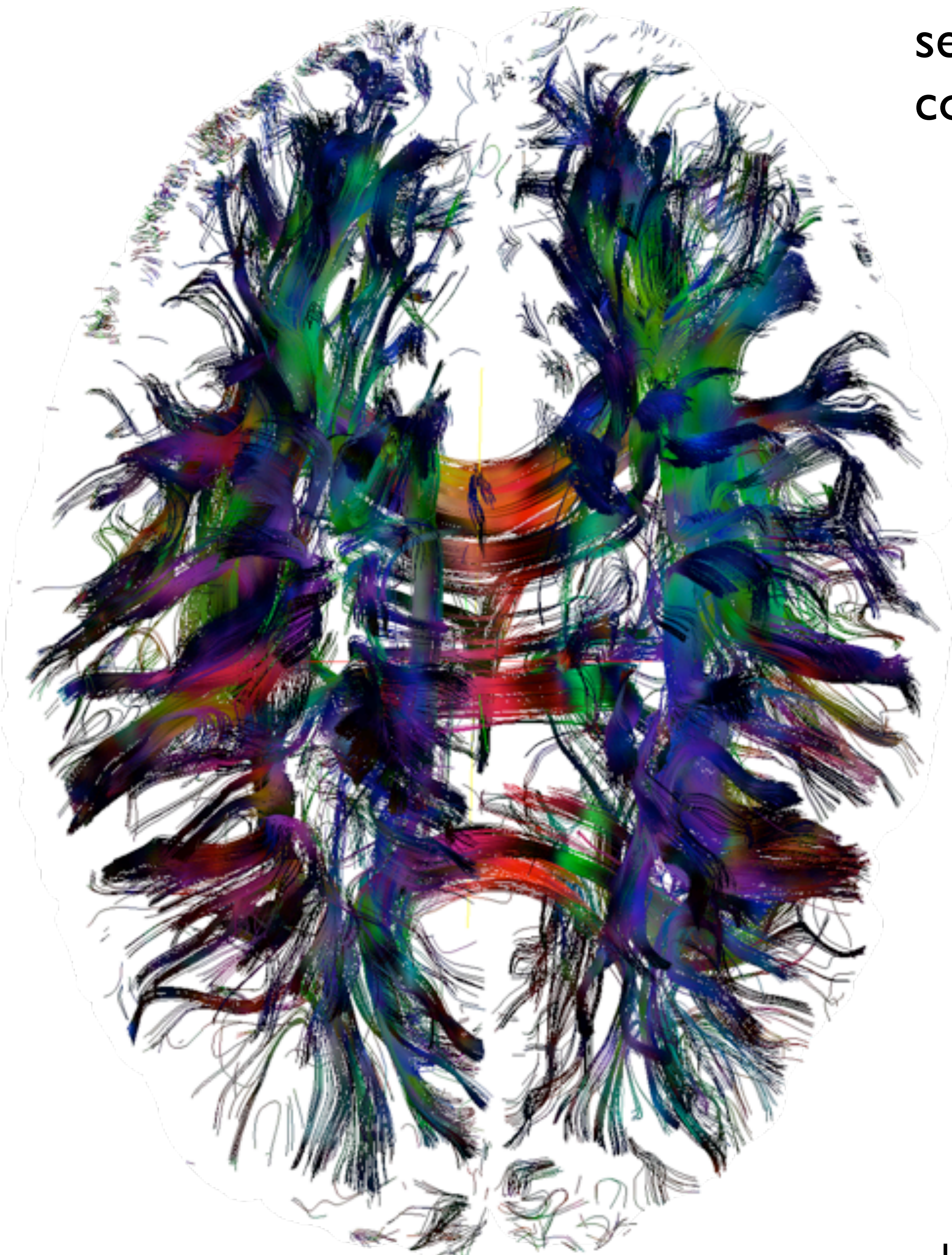
(a little more complicated, in practice ...)



$$\gamma'(t) = v_1(\gamma(t))$$

using method developed by
S Warfield, B Scherrer, Children's Hospital, Harvard

Cortical connectome graph



seed in white matter, retain tracts connecting cortex to cortex

global (tract-based) connectome A^{tr}

$$a_{i,j}^{tr} = \sum_{k \text{ connecting } i \text{ and } j} \frac{1}{\text{length}(k)}$$

local (adjacency-based) connectome A^{loc}

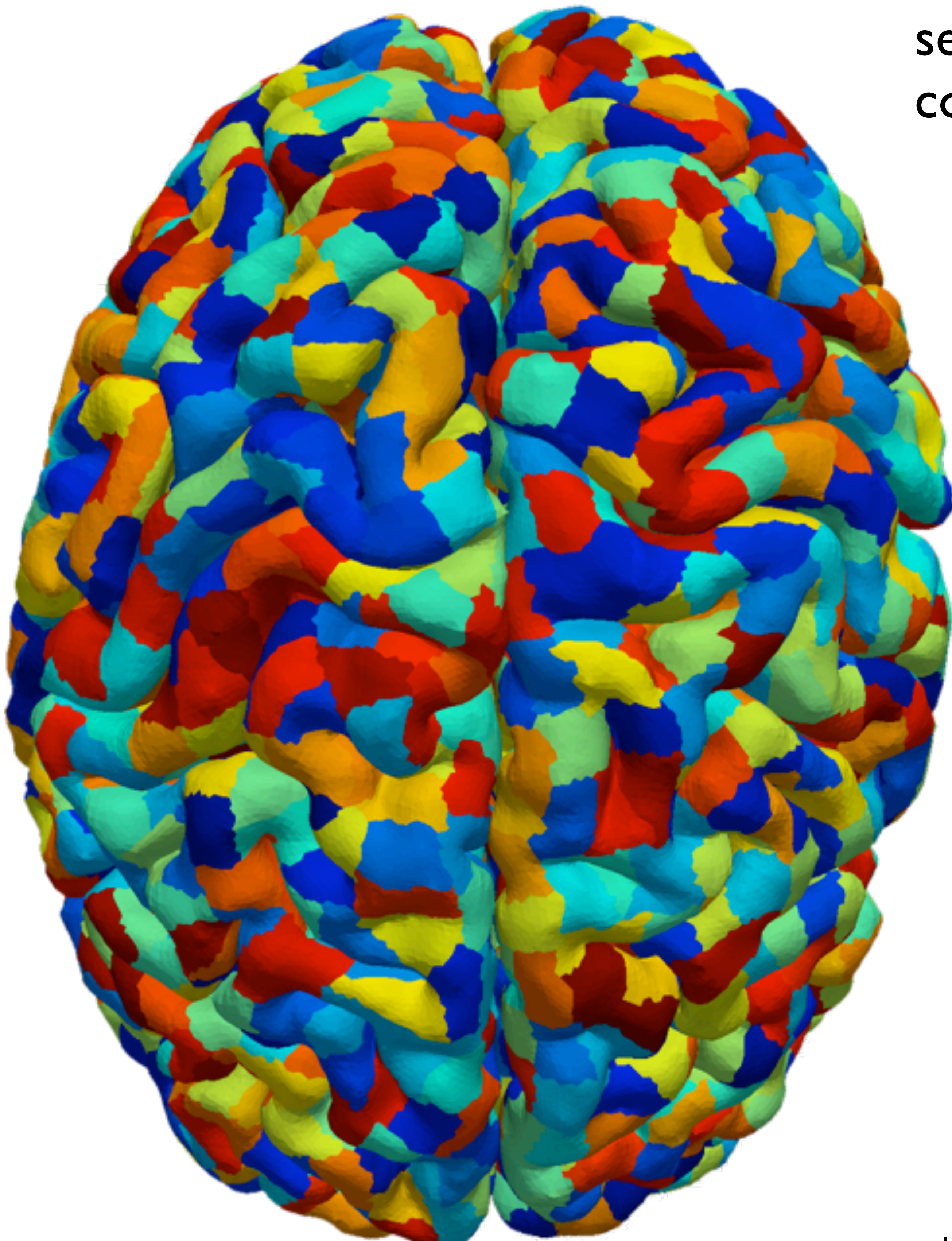
$a_{i,j}^{loc} =$ length of boundary between patches i,j

hybrid connectome A

$$A = \lambda_{tr} A^{tr} + \lambda_{loc} A^{loc}$$

showing 12,124 out of 775,939 cortical-cortical tracts

Cortical connectome graph



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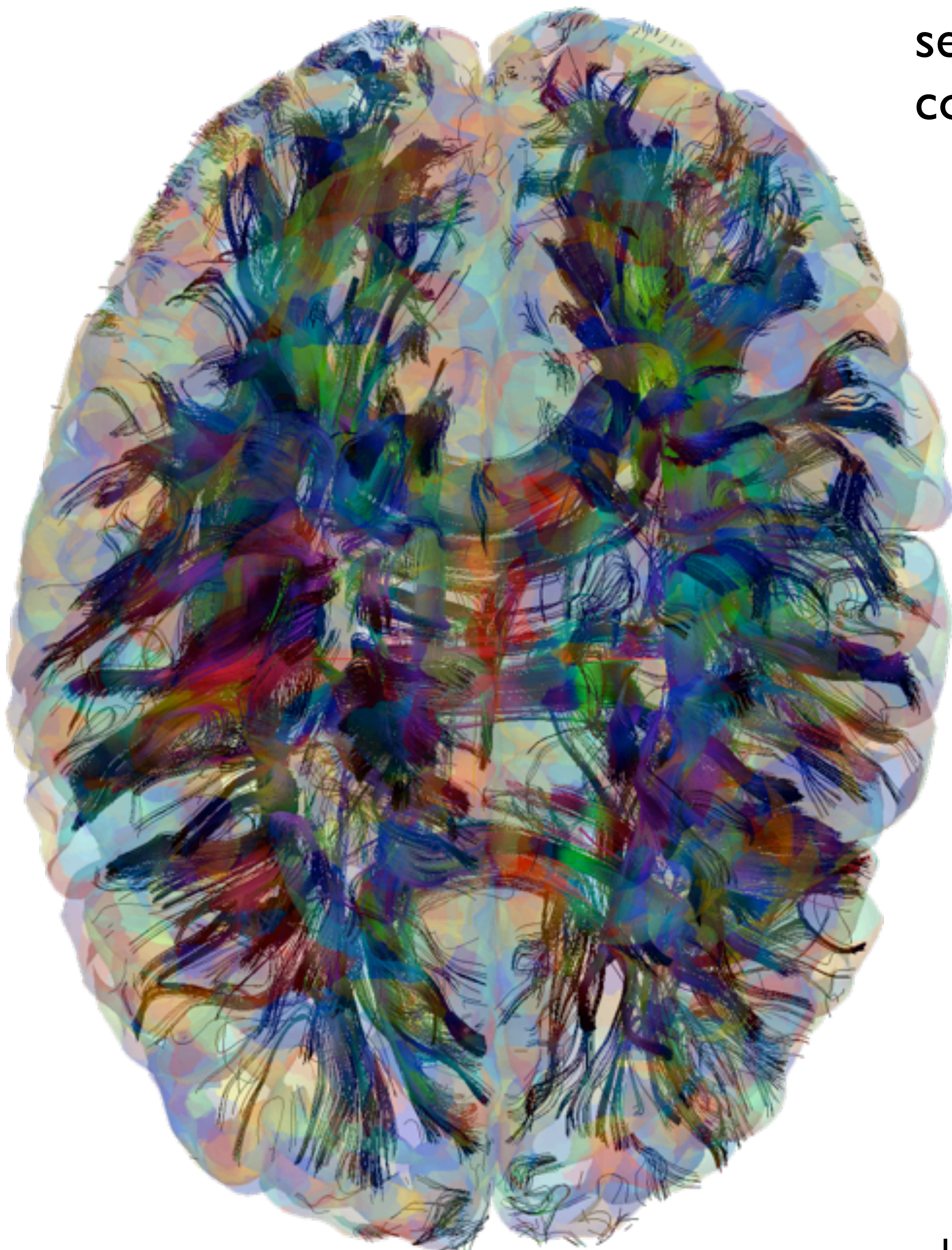

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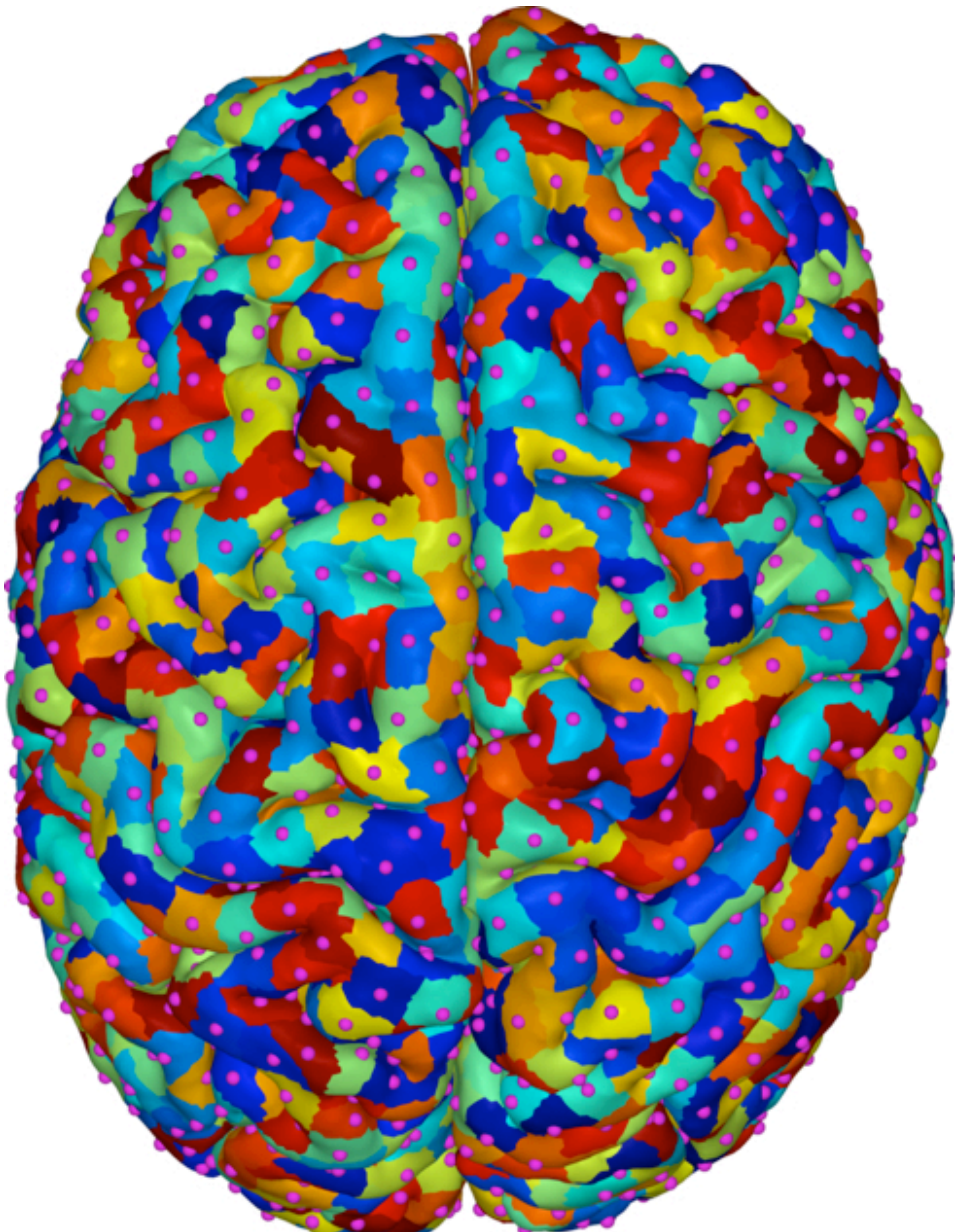
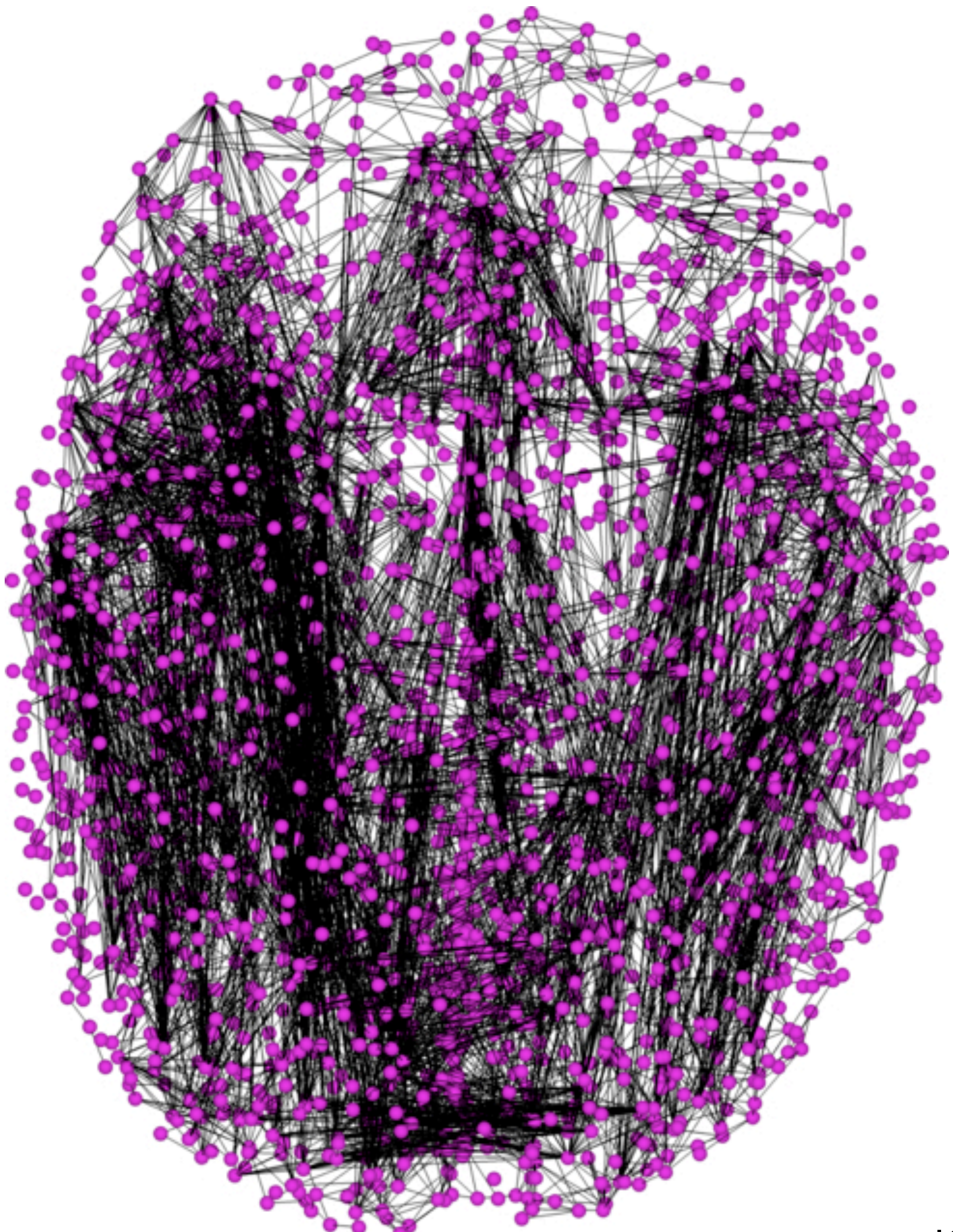
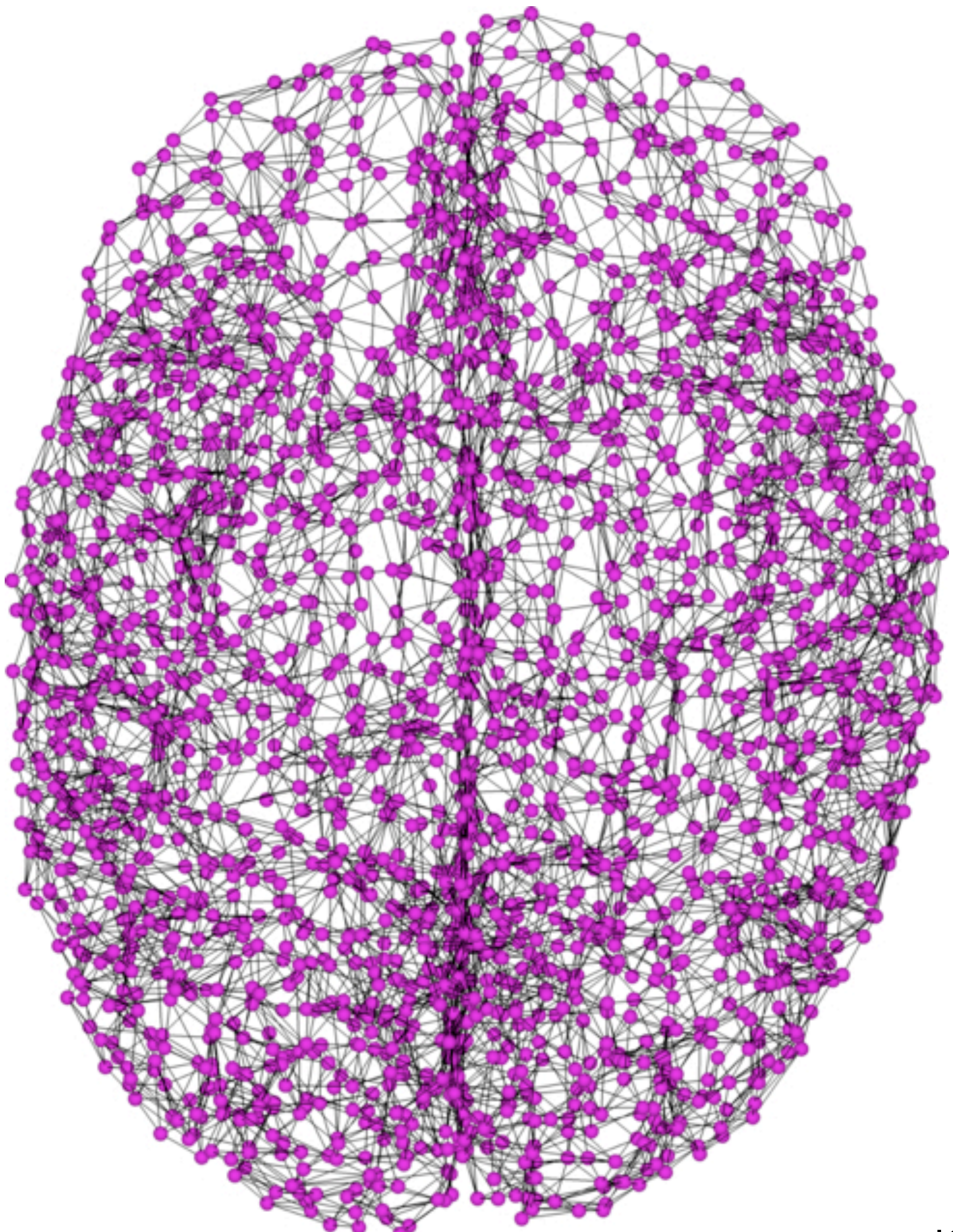
cortex patches, showing patch centers



Global (tract-based) connectome graph



Local connectome graph



EEG inverse problem

linear superposition :

$$\Phi = KJ$$

Inverse problem : Given Φ, find J

N_e equations, N_d unknowns $\Longrightarrow$

infinitely many possibilities for J !

Find J minimizing $\boxed{||\Phi - KJ||^2} + \boxed{f(J)}$

data fidelity prior

$f(J)$: small for "good" J
large for "bad" J

how to build prior using connectivity?

Sparse representation with Cortical Graph Wavelets

Construct SGWT from hybrid connectome A $\quad W : \mathbb{R}^{N_d} \to \mathbb{R}^{KN_d}$

expand $\quad J = \sum_i c_i \psi_i = W^T c$

prior : ℓ_1 penalty on c

$$c^* = \underset{c}{\operatorname{argmin}} \quad ||\Phi - KW^T c||_2^2 + \lambda ||c||_1 \qquad \text{(P)}$$

$$J^* = W^T c^*$$

(P) is convex, L1-LS program

solve with truncated Newton interior point method

Kim, Koh, Lustig, Boyd, Gorinevsky 2007

Motor Potential study

Phan Luu (EGI)

Validation : investigate source estimates for paradigm where we "know" where the sources should be

Experimental paradigm:

Press button (RT,LT,RP,LP)

EEG recording setup :

256 - channel (257 electrodes) sensor net, 250 Hz

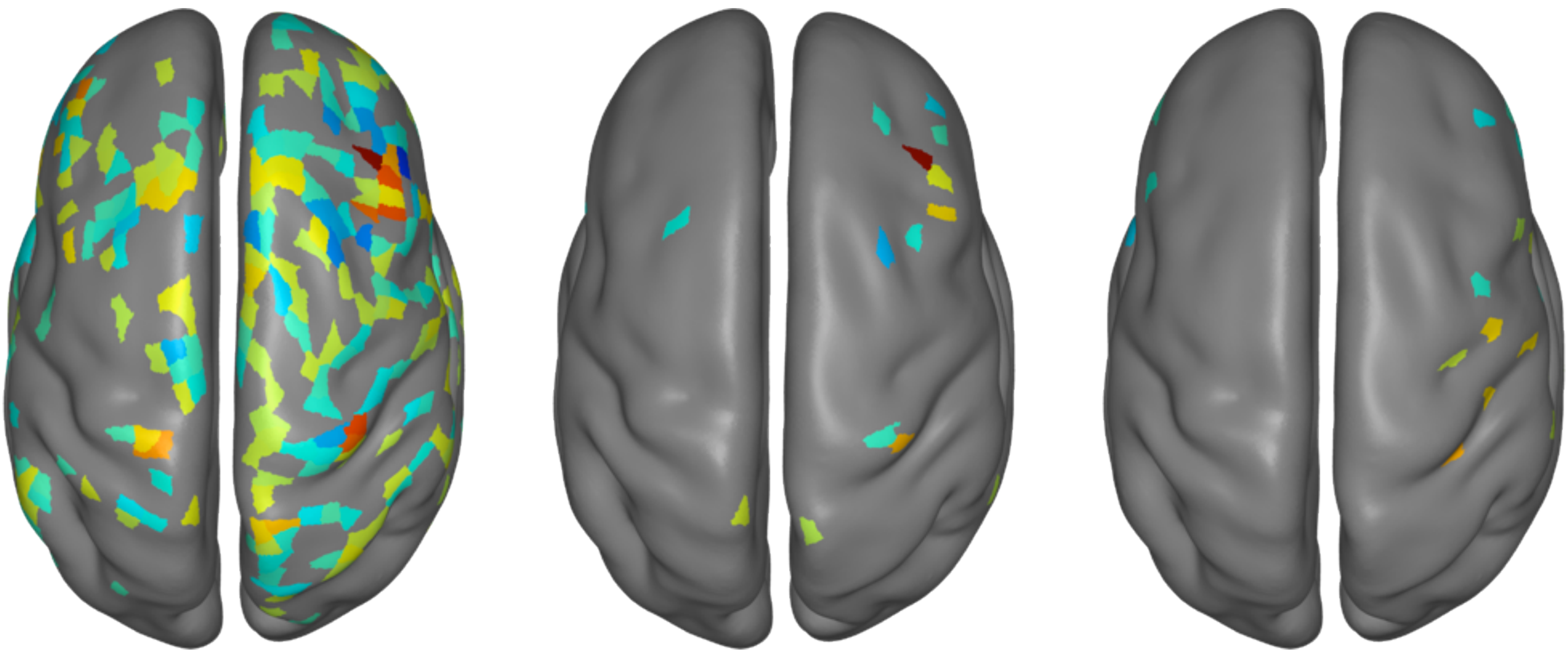
average locked to button press onset (ERP)

expect activation in contralateral motor cortex

Preliminary Results

subject
1470

40 ms before left thumb press



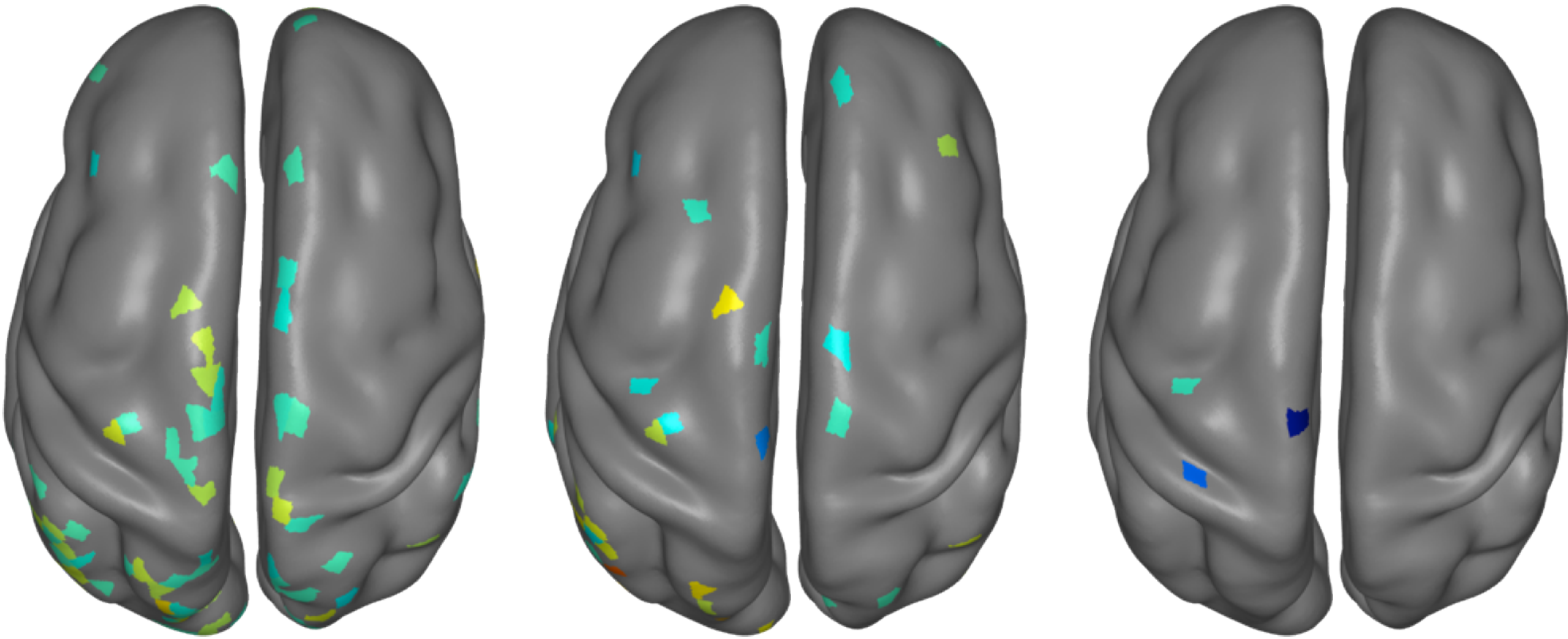
minimum norm
(l_2 penalty)

proposed
(wavelet l_1 penalty)

dipole l_1 penalty

subject
1490

76 ms before right thumb press



minimum norm
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Future Work

Explore parameters (connectome graph construction, SGWT ...)

Sensible automatic selection of regularization parameters (L-curve)

Spatiotemporal estimation via spatiotemporal graph

More appropriate convex solver (path following for λ)?

Alternative convex program formulations

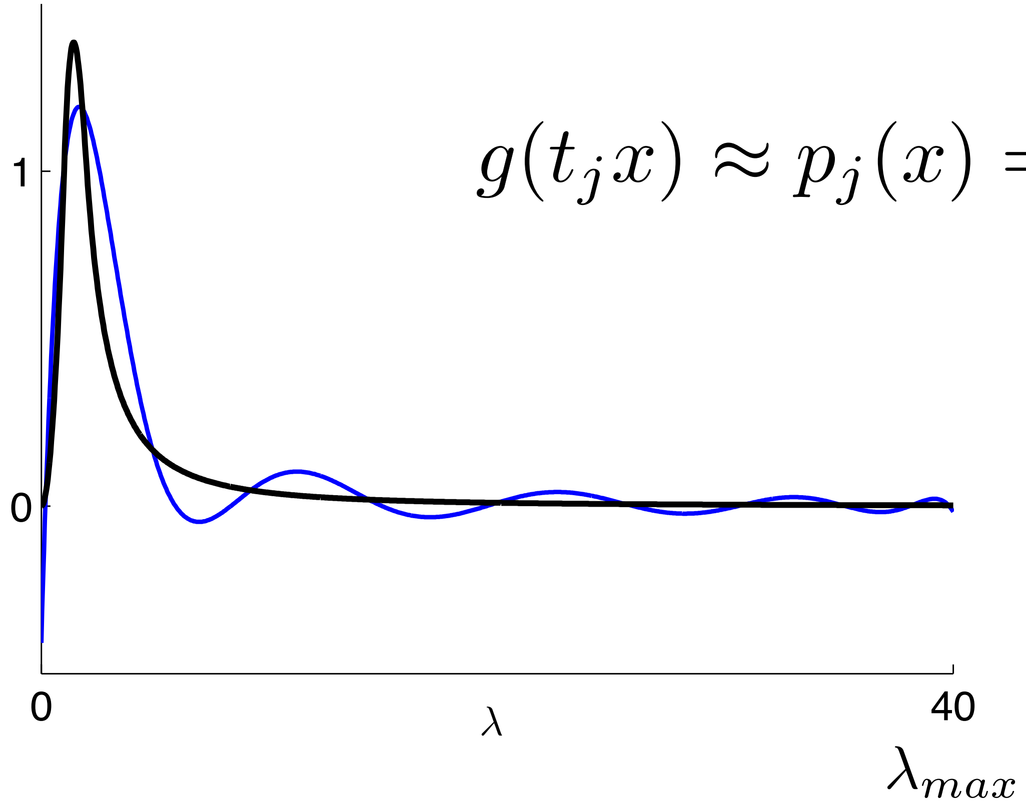
More subjects / different experimental paradigms

Acknowledgements

Allen Malony	NIC, University of Oregon
Don Tucker	EGI / NIC, University of Oregon
Pierre Vandergheynst	EPFL, Switzerland
Rémi Gribonvale	INRIA, France
Adnan Salman	NIC, University of Oregon
Kai Li	NIC, University of Oregon
Simon Warfield	CRL, Children’s Hospital, Harvard
Benoit Scherrer	CRL, Children’s Hospital, Harvard

FIN

Chebyshev polynomials for fast SGWT



$$g(t_j x) \approx p_j(x) = \sum_{k=0}^{M} c_{j,k} T_k(x)$$

Chebyshev approx
for M=10

$$g(t_j L) f \approx p_j(L) f = \sum_{k=0}^{M} c_{j,k} (T_k(L) f)$$

$$T_k(x) = 2x T_{k-1}(x) - T_{k-2}(x)$$

$$T_k(L) f = 2L(T_{k-1}(L) f) - T_{k-2}(L) f$$

computing $T_k(L) f$ is fast if L is *sparse*

Some Cortical Graph Wavelets