Non-rigid Manifolds Matching via Geometric Modeling Methods

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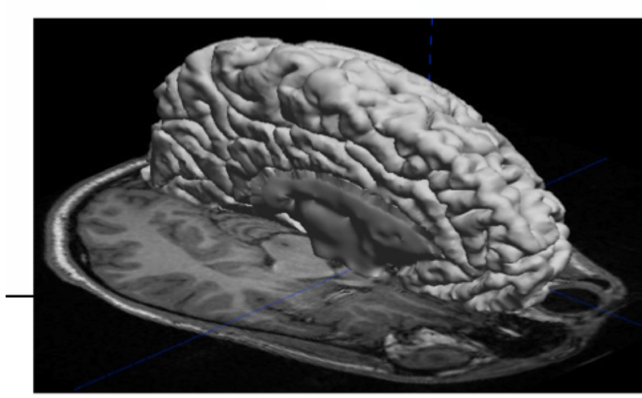
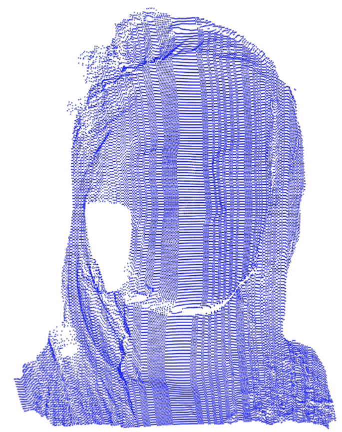
Manifold-structured data in 3D

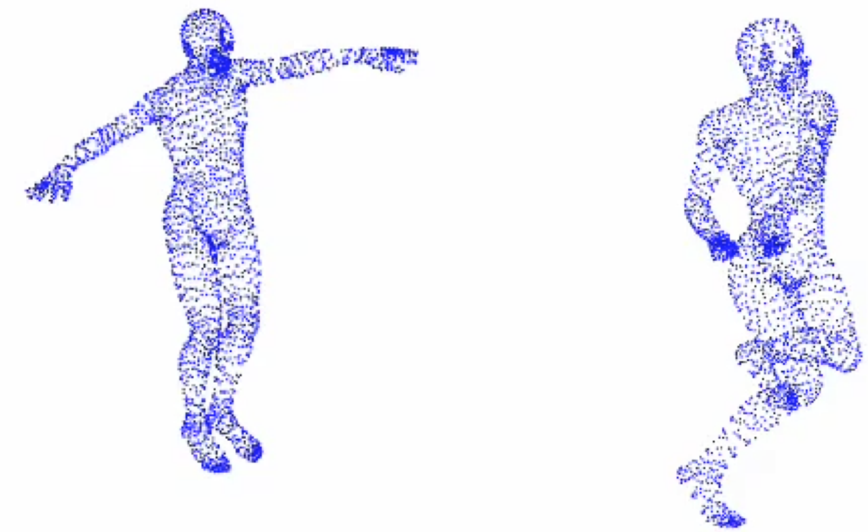
- 3D modeling
- Image Processing
- Medical Imaging
- ....

Magnetic Resonance scanner

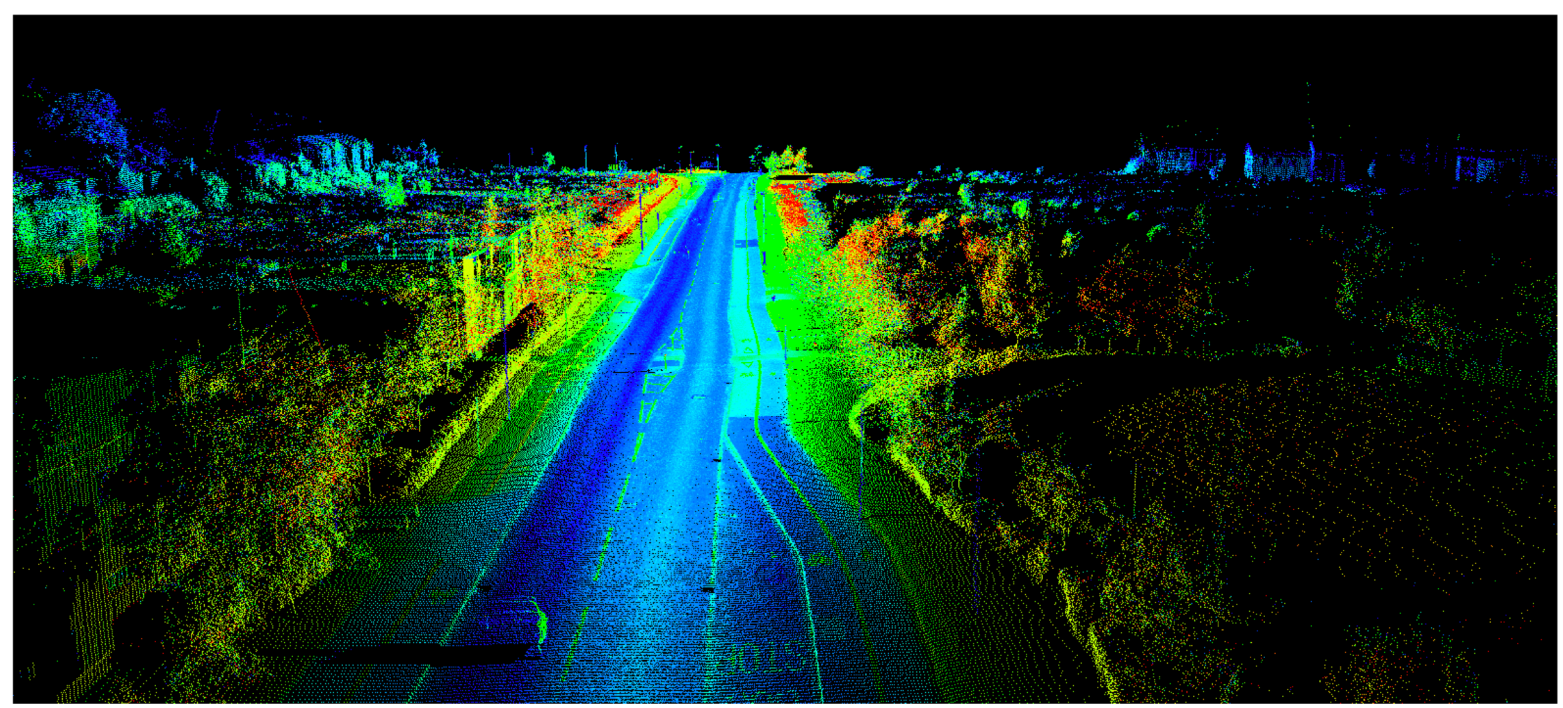




Data from XYZT Lab



Intrinsic comparison
(Data from TOSCA)



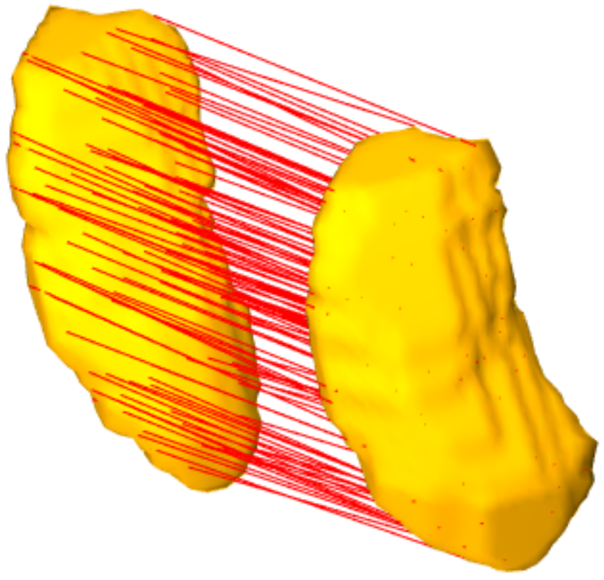
Nonrigid manifold matching: Challenges

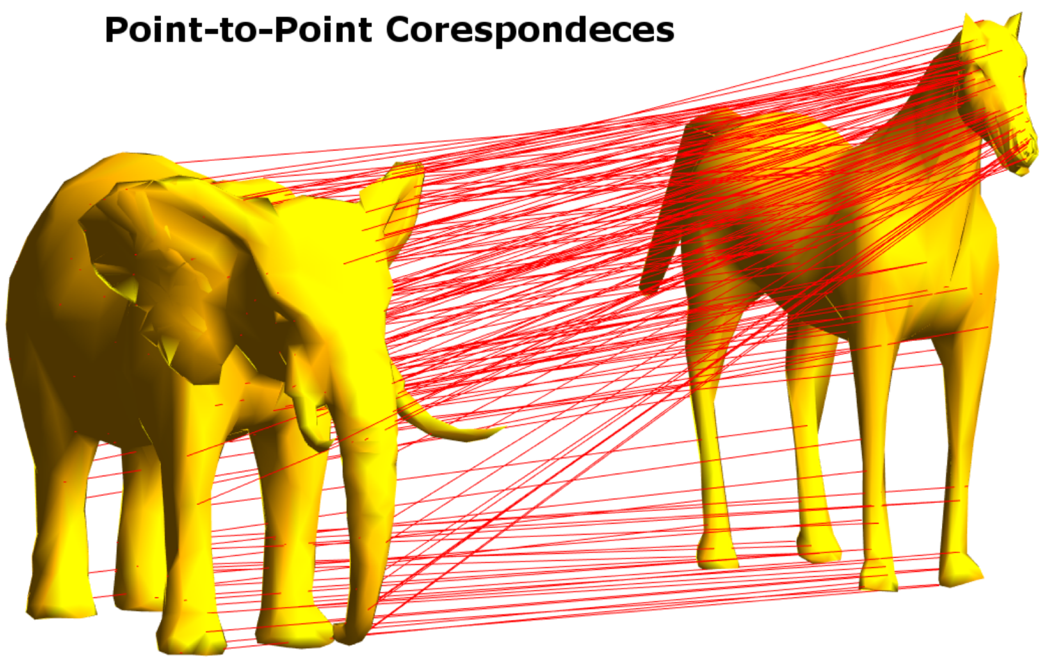
- **Extrinsic: Different poses (each person has 10 poses)**

FAUST

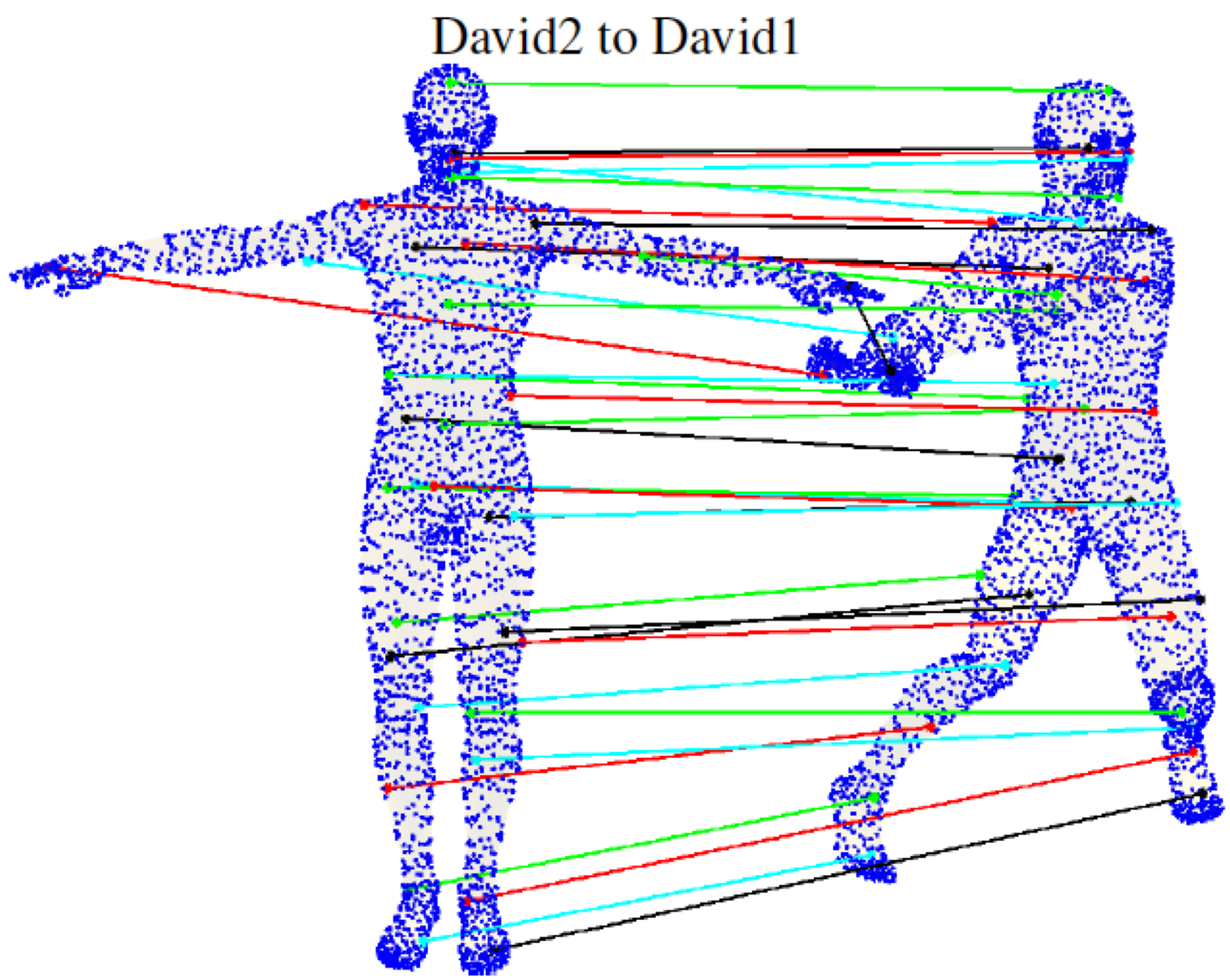
- **Intrinsic: Different metrics**

Compute Invariant Features!





Isometric (nearly isometric) shape matching



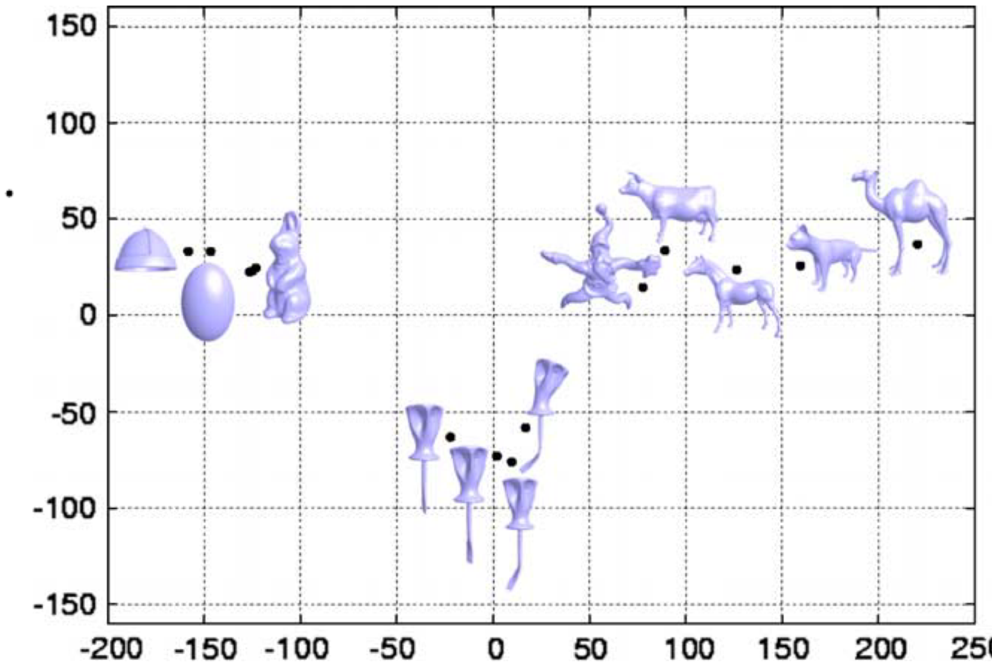
Laplace-Beltrami operator: A Bridge from Local to Global

Given a d-dimensional manifold $(\mathcal{M}, g)$,

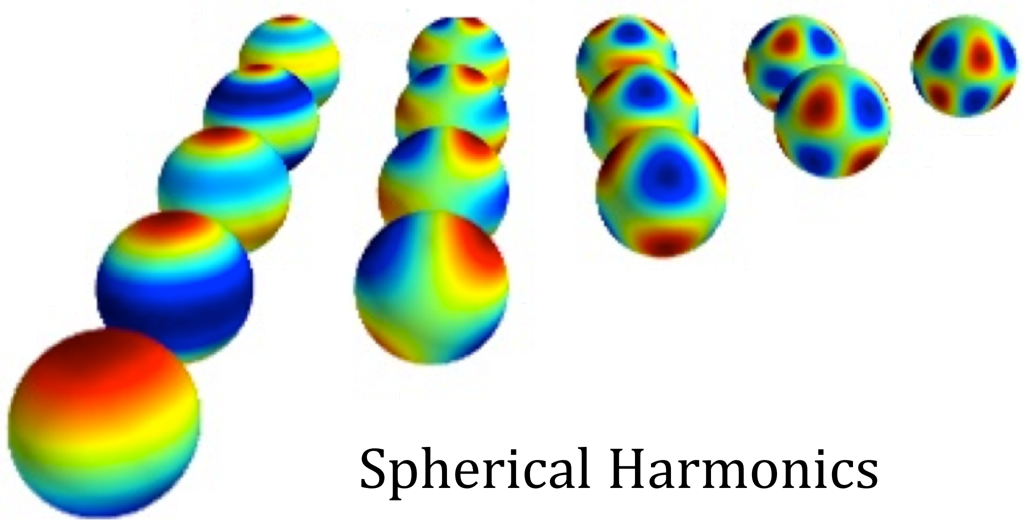
$$-\triangle_{\mathcal{M}}\phi_n = -\frac{1}{\sqrt{G}}\frac{\partial}{\partial x_i}(\sqrt{G}g^{ij}\frac{\partial \phi}{\partial x_j}) = \lambda_n \phi_n, \;\; n = 0, 1, 2, \cdots$$

$$\Longleftrightarrow$$

$$\sum_i \int_{\mathcal{M}} ||\nabla_{\mathcal{M}}\phi_i||^2 d\mathcal{M} \quad \text{s.t.} \int_{\mathcal{M}} \phi_i \phi_j \mathrm{d}vol_{g_{\mathcal{M}}} = \delta_{ij}$$



Shape DNA **[M. Reuter et al]**



Spherical Harmonics

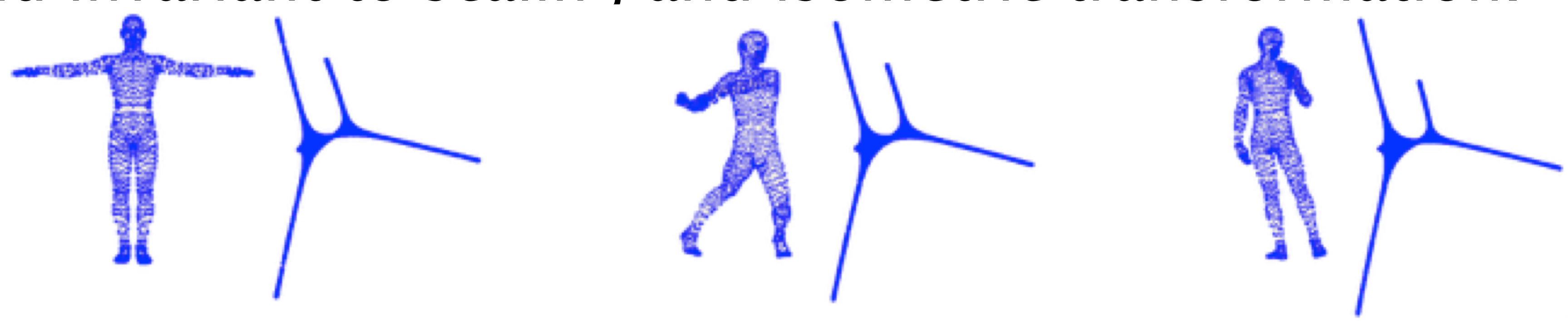
Laplace-Beltrami Eigen embedding

$$I_{\mathcal{M}}^{\Phi} : \mathcal{M} \to \mathbb{R}^{\infty}, I_{\mathcal{M}}^{\Phi}(x) = \left(\frac{\phi_1(x)}{\lambda_1^{d/4}}, \cdots, \frac{\phi_n(x)}{\lambda_n^{d/4}}, \cdots \right)^T$$

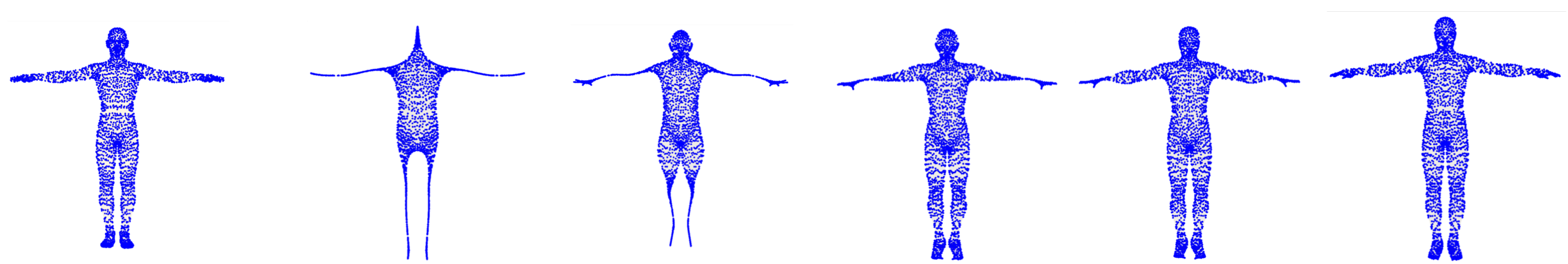
[GPS, Rustamov]

Nice properties for LB eigenembedding:

- Intrinsic and invariant to scaling and isometric transformation.



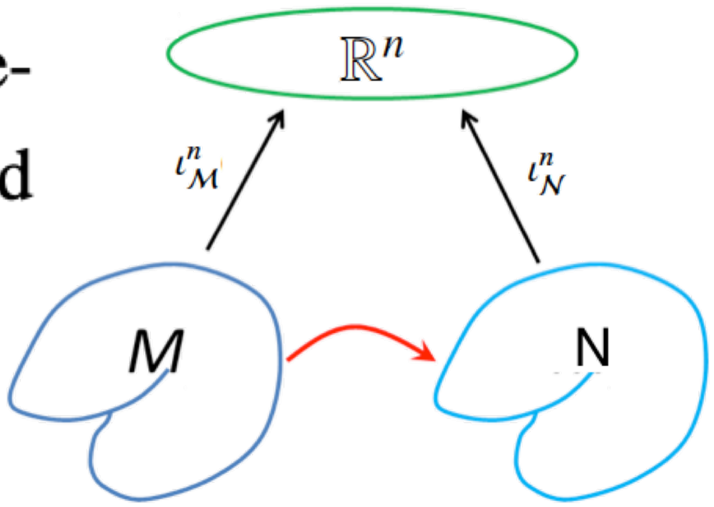
- A natural multiscale characterization with global information. A good candidate for nonlinear dimension reduction for point clouds in higher dimensions.



Original **n=20** **n=50** **n=100** **n=150** **n=200**

Registration in embedding space

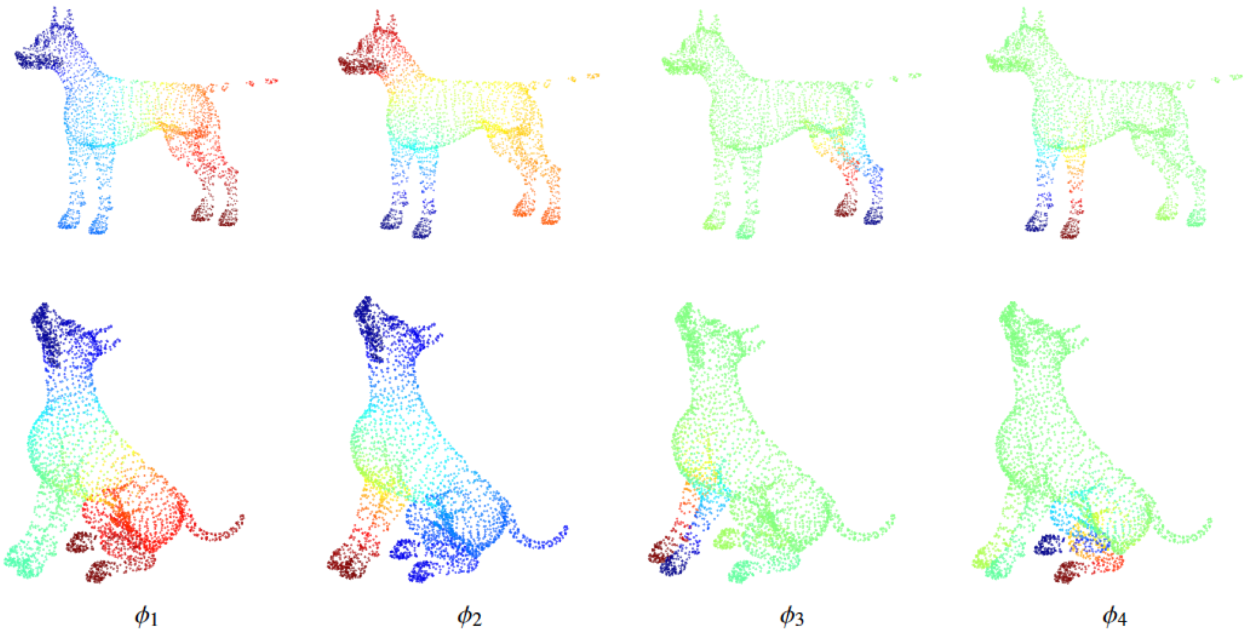
Let $(\mathcal{M}, g_{\mathcal{M}})$ and $(\mathcal{N}, g_{\mathcal{N}})$ be two isometric d-dimensional Riemannian manifolds with LB eigensystem $\Phi_n = \{\lambda_k, \phi_k\}_{k=1}^n$ and $\Psi_n = \{\eta_k, \psi_k\}_{k=1}^n$ respectively.



$$\iota_{\mathcal{M}}^n : \mathcal{M} \to \mathbb{R}^n, \qquad u \mapsto p = \iota_{\mathcal{M}}^n(u) = \left(\frac{\phi_1(u)}{\lambda_1^{d/4}}, \cdots, \frac{\phi_n(u)}{\lambda_n^{d/4}} \right)$$

Challenges:

- Ambiguities from LB eigenmap;
- Different number of points;
- Different density distributions.



Spectral l2-distance [Lai-Shi-Toga-Chan'10]

Definition 1 (spectral l^2-distance) *Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two surfaces. We define*

$$
\begin{aligned}
d_{\Phi_1}^{\Phi_2}(x, \mathcal{M}_2) &= \inf_{y\in\mathcal{M}_2} ||I_{\mathcal{M}_1}^{\Phi_1}(x) - I_{\mathcal{M}_2}^{\Phi_2}(y)||_2 \ , \ \forall \ x \in \mathcal{M}_1, \\
d_{\Phi_1}^{\Phi_2}(\mathcal{M}_1, y) &= \inf_{x\in\mathcal{M}_1} ||I_{\mathcal{M}_1}^{\Phi_1}(x) - I_{\mathcal{M}_2}^{\Phi_2}(y)||_2 \ , \ \forall \ y \in \mathcal{M}_2.
\end{aligned}
$$

The spectral l^2-distance $\mathrm{d}(\mathcal{M}_1, \mathcal{M}_2)$ between $\mathcal{M}_1$ and $\mathcal{M}_2$ independent of the choice of eigen-systems is then defined as:

$$
d(\mathcal{M}_1, \mathcal{M}_2) = \inf_{\Phi\in\mathcal{B}(\mathcal{M}_1), \Phi_2\in\mathcal{B}(\mathcal{M}_2)} \max\left\{ \int_{\mathcal{M}_1} d_{\Phi}^{\Phi_2}(x, \mathcal{M}_2)\mathrm{d}_{\mathcal{M}_1}(x) \, , \int_{\mathcal{M}_2} d_{\Phi}^{\Phi_2}(\mathcal{M}_1, y)\mathrm{d}_{\mathcal{M}_2}(y) \right\}.
$$

Theorem 1 *Let $\mathfrak{S}$ be the set of surfaces with normalized area. For any surfaces $\mathcal{M}, \mathcal{M}', \mathcal{N} \in \mathfrak{S}$,*

1. $d(\mathcal{M}, \mathcal{M}') = d(\mathcal{M}', \mathcal{M})$,
2. $d(\mathcal{M}, \mathcal{M}') \leq d(\mathcal{M}, \mathcal{N}) + d(\mathcal{N}, \mathcal{M}')$
3. $d(\mathcal{M}, \mathcal{M}') = 0 \iff \mathcal{M}$ *is isometric to* $\mathcal{M}'$

Therefore, the spectral l^2-distance is a rigorous distance on the shape space $\mathfrak{S}$.

Rotation invariant sliced Wasserstein distance [Lai-Zhao'17]

Inspired by the Sliced-Wasserstein distance [Rabin-Peyre-Delon-Bernot'12], we propose the following rotation-invariant sliced-Wasserstein distance for registration

$$\mathrm{RSWD}\Big((\mathcal{P},\mu^{\mathcal{P}}),(\mathcal{Q},\mu^{\mathcal{Q}})\Big)^2 = \min_{R\in O(n)} \int_{S^{n-1}} \min_{\sigma\in\mathrm{ADM}(\pi_{\#}^{\theta,R}\mu^{\mathcal{P}},\pi_{\#}^{\theta}\mu^{\mathcal{Q}})} \int_{\mathbb{R}\times\mathbb{R}} \|x-y\|_2^2 \,\mathrm{d}\sigma(x,y)\ \mathrm{d}\theta$$

Theorem (Lai-Zhao). RSWD$(\cdot,\cdot)$ *defines a distance on the space* $\mathfrak{M}_n/\sim$.

Iterative method for computing Robust Sliced-Wasserstein distance

Initialize $R^0, \sigma^0(\theta)$,

while *"not converge"* **do**

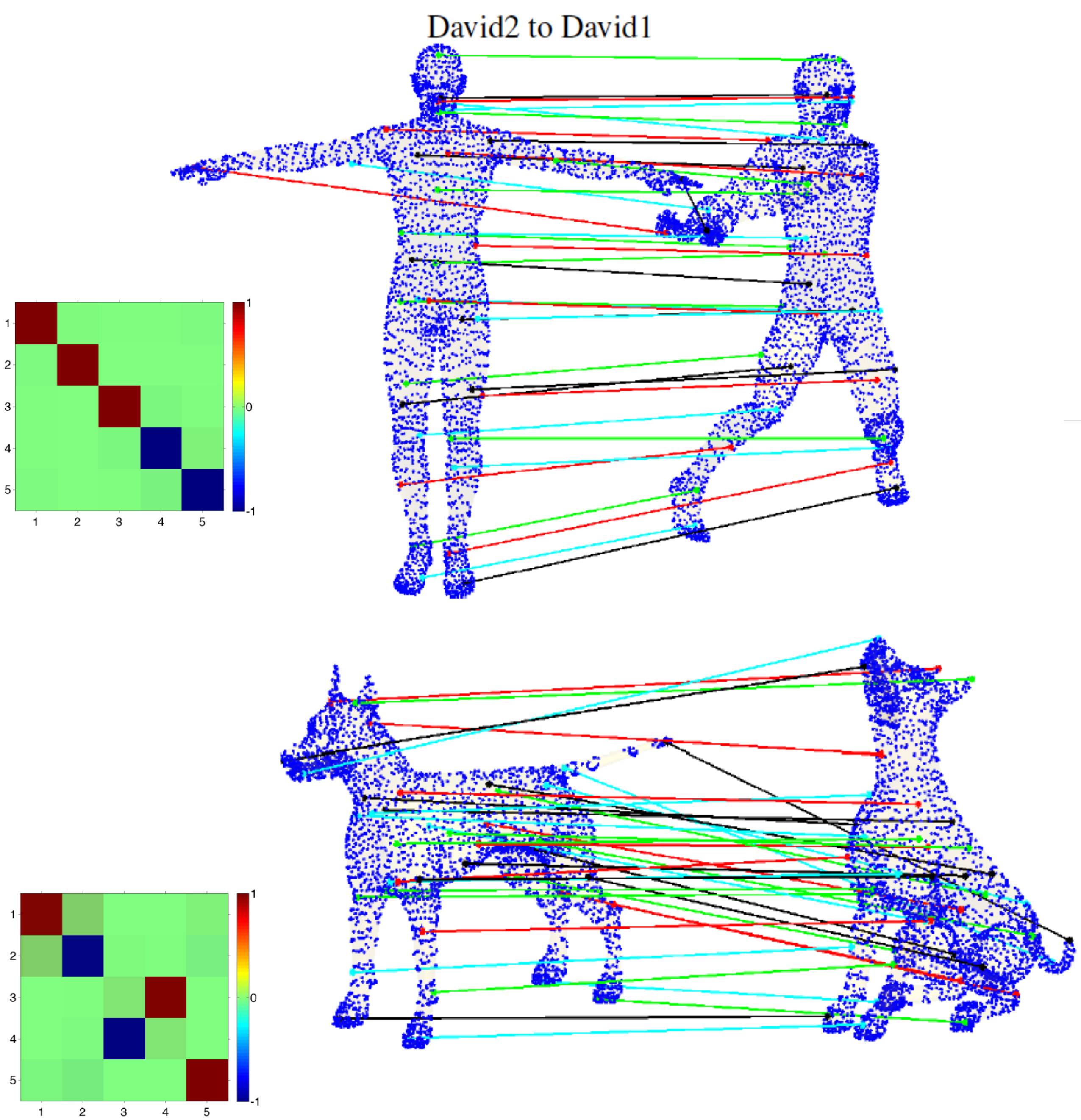
1. $R^k = \min_{R\in O(n)} \int_{S^{n-1}} \sum_{i,j} \sigma_{i,j}^{k-1}(\theta)\Big(p_i R\theta^T - q_j\theta^T\Big)^2 \,\mathrm{d}\theta.$

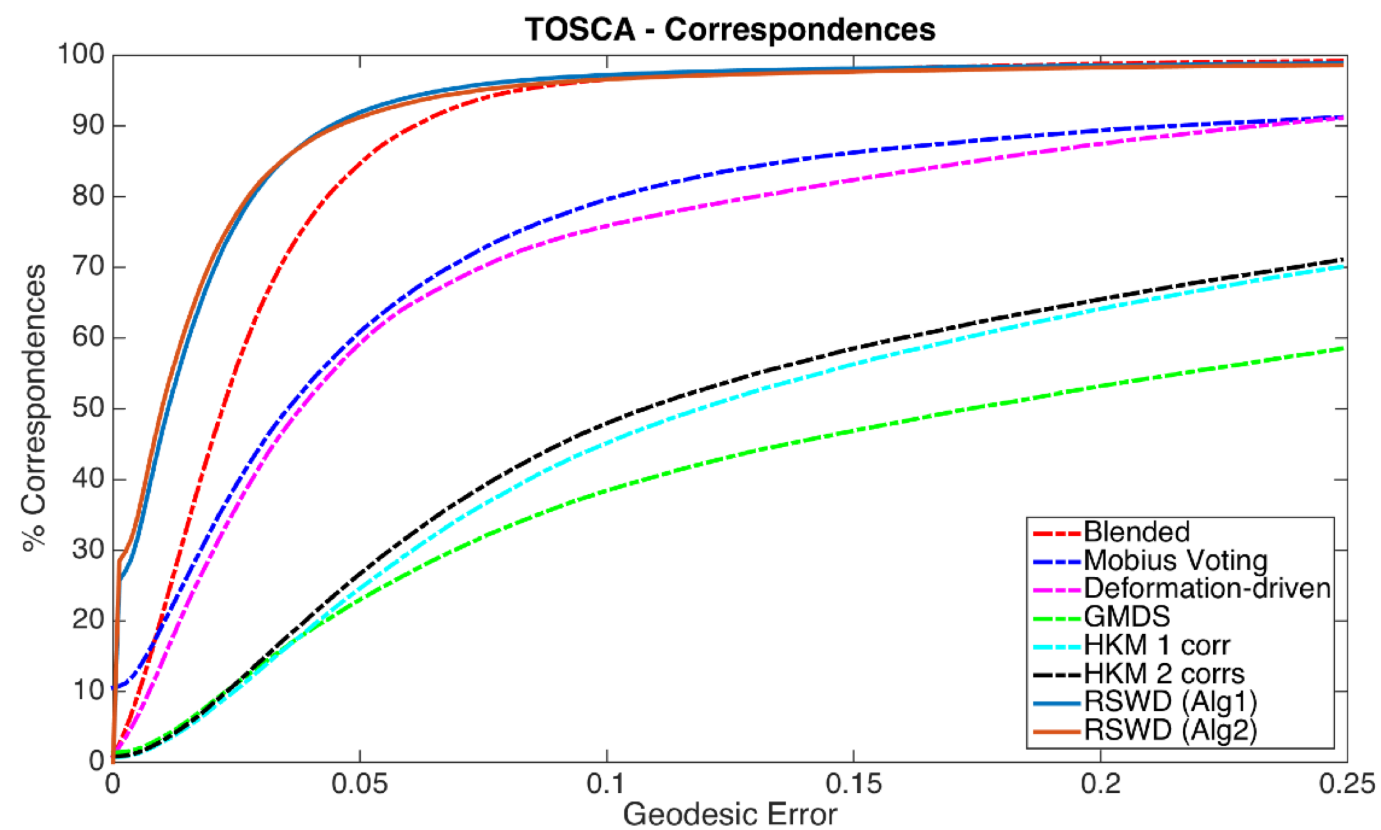
2. For each $\theta \in S^{n-1}$, solve $\sigma^k(\theta) = \min_{\sigma\in\mathrm{ADM}(\mu_{\theta,R}^P,\mu_{\theta}^Q)} \sum_{i,j} \sigma_{ij}\Big(p_i R^k\theta^T - q_j\theta^T\Big)^2$ by sorting.

Theorem. *Let* $E(R) = \int_{\theta\in\mathbb{R}^n,\|\theta\|=1} \min_{\sigma\in\mathrm{ADM}(\mu_{\theta,R}^P,\mu_{\theta}^Q)} \sum_{i,j} \sigma_{ij}\Big(p_i R\theta^T - q_j\theta^T\Big)^2 \,\mathrm{d}\theta$ *and* $\{R^k\}_{k=1}^{\infty}$ *be a sequence generated by the above algorithm, then* $E(R^{k+1}) \leq E(R^k)$.

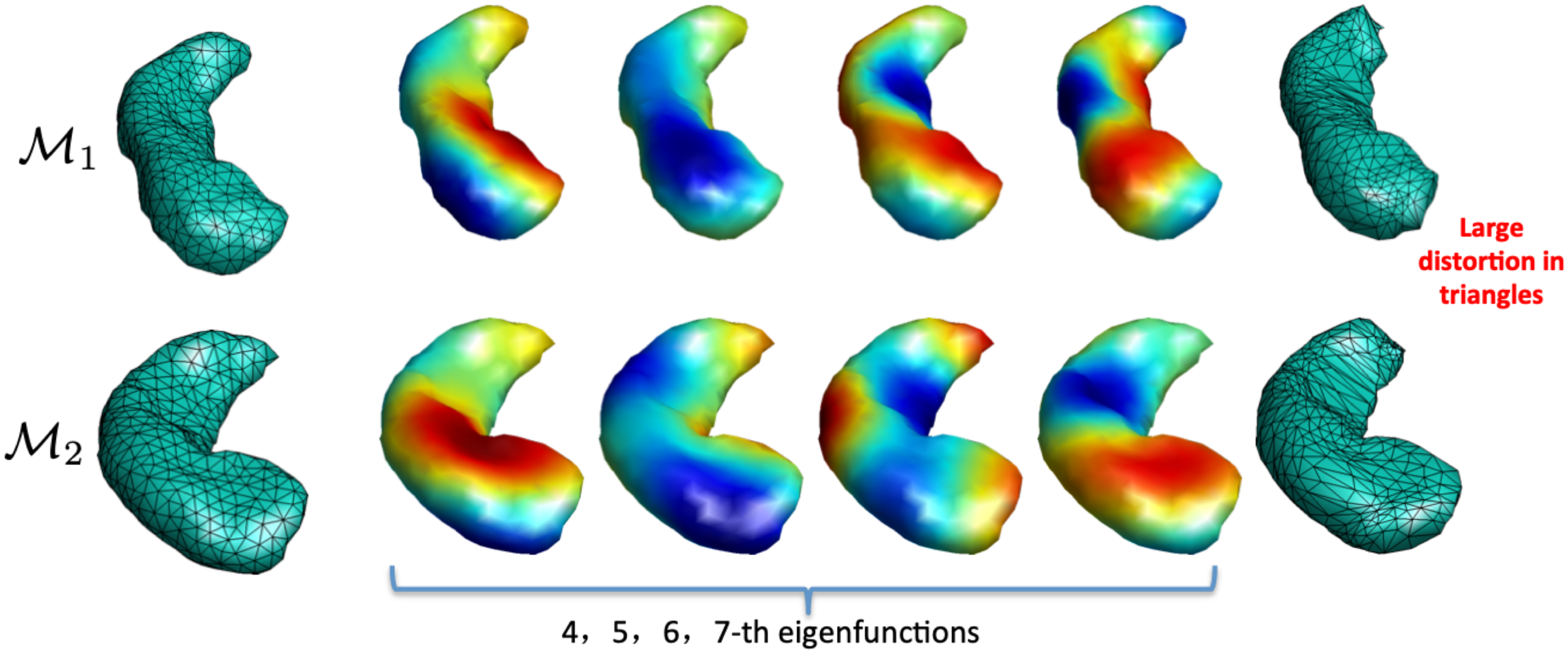
Intrinsic comparisons using LB eigenmaps + optimal transportation [Lai-Zhao'17]

$$\mathrm{RSWD}\Big((\mathcal{P},\mu^{\mathcal{P}}),(\mathcal{Q},\mu^{\mathcal{Q}})\Big)^2 = \min_{R\in O(n)} \int_{S^{n-1}} \min_{\sigma\in \mathrm{ADM}(\pi_{\#}^{\theta,R}\mu^{\mathcal{P}},\pi_{\#}^{\theta}\mu^{\mathcal{Q}})} \int_{\mathbb{R}\times\mathbb{R}} \|x-y\|_2^2 \, d\sigma(x,y) \; d\theta$$

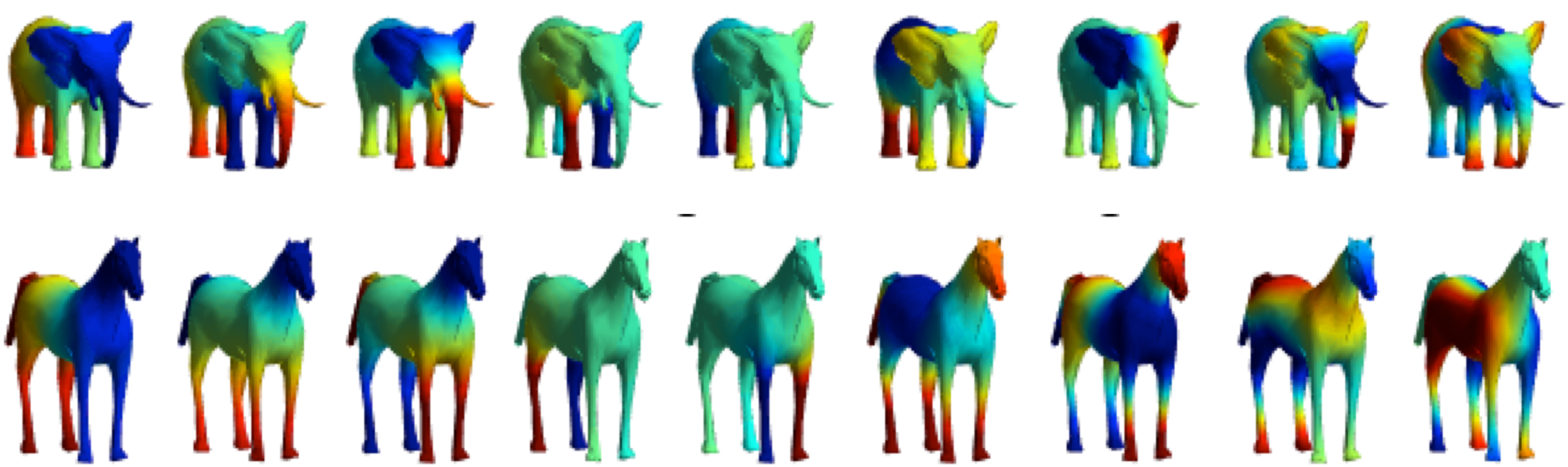




Non-isometric manifolds



Deform manifolds to make them isometric.



Challenges:

1. Do not know the amount of deformation without knowing correspondence
2. The eigesystem of the deformed manifold relies on the reconstruction of the manifold
3. Ambiguities of the eigensystem
4. Representation of deformation (characterization of the shape space)

Extrinsic representation V.S. Intrinsic representation

- **Extrinsic representation**

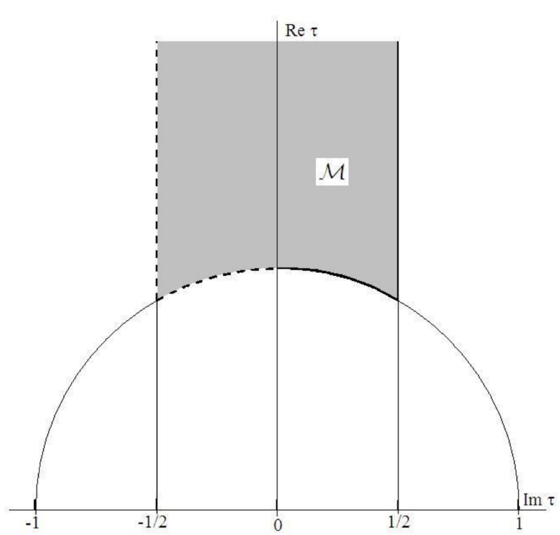
  Use the embedding coordinates (x, y, z). Hard to preserve diffeomorphism

Intrinsic representation: Genus-0 surfaces are conformally equivalent

$$\text{Fix a genus-0 surface } (\mathcal{M}, g).$$
$$\text{Shape space} = \{(\mathcal{M}, wg) \mid w > 0\}.$$

- **Intrinsic representation: high genus surfaces**

$$\text{Moduli space} \times \text{conformal class}$$



Genus 1

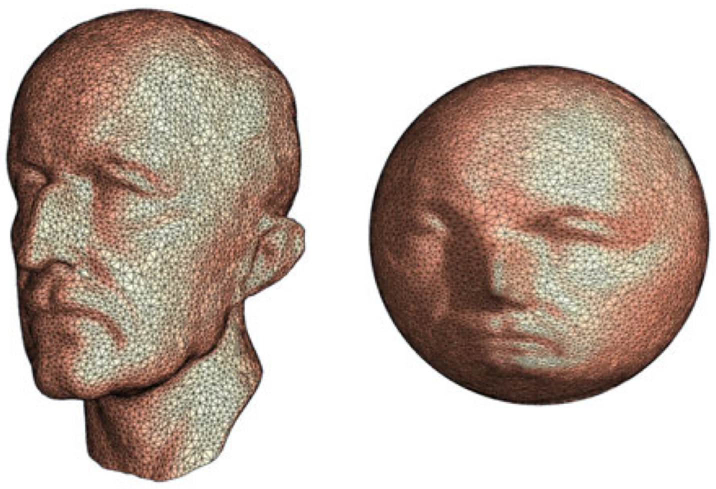
$$\begin{array}{ccc} \mathfrak{T}_g & \sim & \mathbb{R}^{6g-6} \\ \downarrow & & \\ \mathfrak{M}_g & & \end{array}$$

Genus g>1

LB eigensystem via conformal deformation

- A conformal map is one which preserves angles locally. Formally, it preserves the first fundamental form up to a positive scaling factor.

  The map $F : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$ is conformal if and only if the pull back metric has: $F^*(g_2) = w^2 g_1$, where w^2 is positive. It is well known that any two genus zero surfaces are conformal to each other.



- **Proposition:** Let $\{\phi_n^{w^2}, \lambda_n^{w^2}\}_n$ be a LB eigensystem of a conformally deformed surface $(\mathcal{M}, w^2 g_{\mathcal{M}})$, then $\{\phi_n^{w^2}, \lambda_n^{w^2}\}_n$ is equivalent to the following weighted LB eigensystem on $(\mathcal{M}, g_{\mathcal{M}})$:

$$\begin{aligned} -\Delta_g \phi_n &= \lambda w^2 \phi_n, \; n = 0, 1, \ldots, \\ \textstyle\int_{\mathcal{M}} \phi_i \phi_j w^2 \; \mathrm{d}vol_{g_{\mathcal{M}}} &= \delta_{ij} \end{aligned} \tag{1}$$

Or it can be written as the following variational formula

$$\sum_i \int_{\mathcal{M}} ||\Delta_{\mathcal{M}} \phi_i||^2 d\mathcal{M} \quad \text{s.t.} \int_M \phi_i \phi_j w^2 \mathrm{d}vol_{g_{\mathcal{M}}} = \delta_{ij}$$

Unsupervised: Nonisometric mapping via conformal deformation [Shi-Lai-Toga' 11, 14.]

$$(\omega_1^*, \omega_2^*) = \arg\min_{\omega_1,\omega_2} \frac{1}{S_1} D(\mathcal{M}_1^{\omega}, \mathcal{M}_2) + \frac{1}{S_2} D(\mathcal{M}_1, \mathcal{M}_2^{\omega_2}) + \xi \sum_{i=1,2} \int_{\mathcal{M}_i} \|\nabla_{\mathcal{M}_i} \omega_i\|^2$$

where conformaldeformation of $(\mathcal{M}_i, g_i)$ is denoted by $\mathcal{M}_i^{\omega_i} = (\mathcal{M}_i, \omega_i g_i)$.

Proposition (Variations for LB eigen-system w.r.t. conformal deformation)

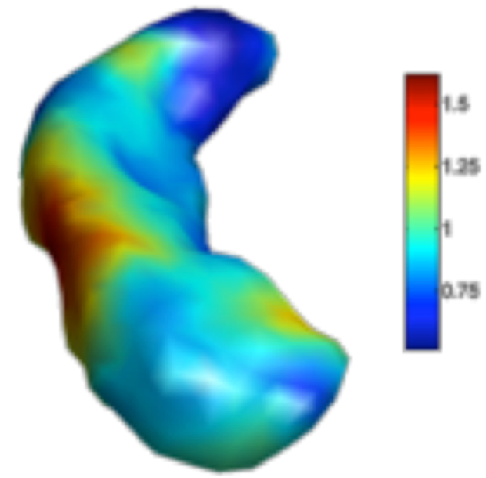
- $\Delta_{\mathcal{M}^\omega} \phi_k = -\lambda_k \phi_k \Longleftrightarrow \Delta_{\mathcal{M}} \phi_k = -\omega \lambda_k \phi_k$.
- *Let (λ_n, ϕ_k) be a simple eigenpair of $-\Delta_{\mathcal{M}^\omega}$. The variation of λ_k, ϕ_k w.r.t. a perturbation of ω are given by:*

$$\frac{\delta \lambda_k}{\delta \omega} = -\lambda_k \frac{(\omega^{-1}\phi_k, \phi_k)_{\omega g}}{(\phi_k, \phi_k)_{\omega g}}$$

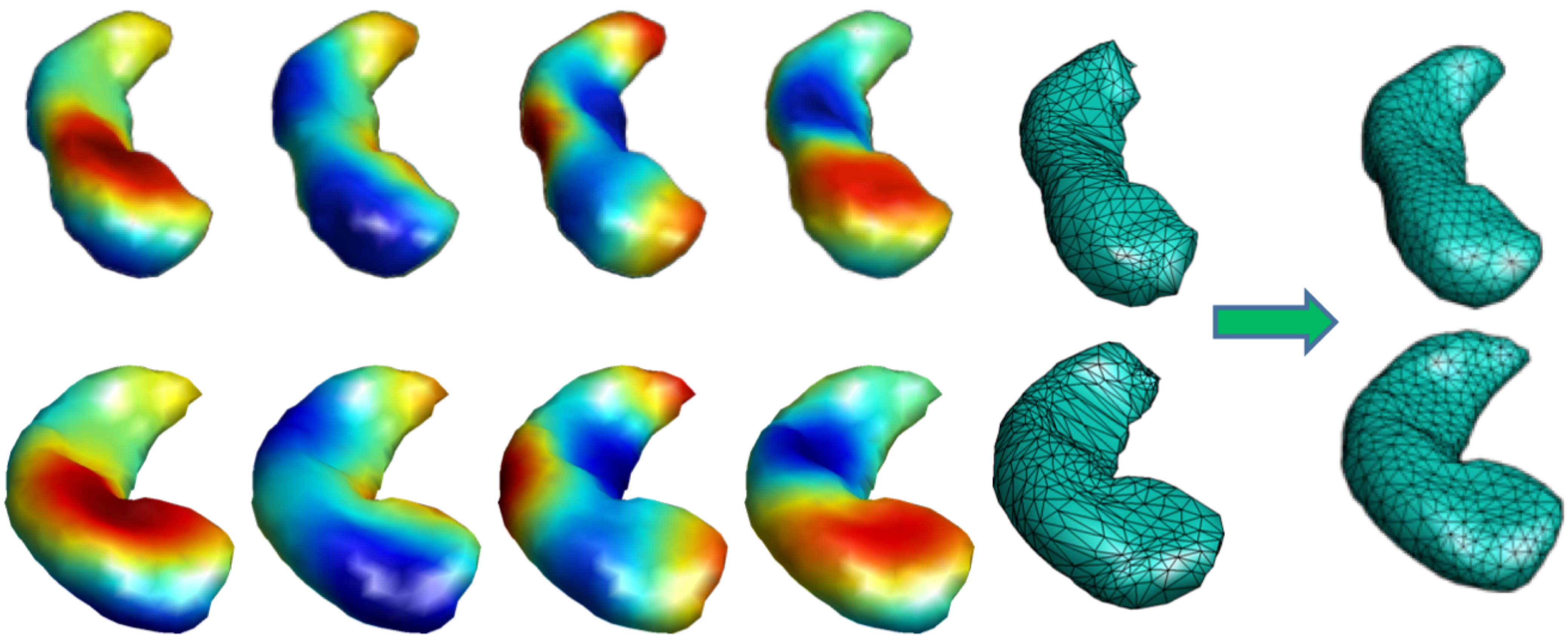
$$(\Delta_{\mathcal{M}^\omega} + \lambda_k) \frac{\delta \phi}{\delta \omega} = -\frac{\delta \lambda_k}{\delta \omega} \phi_k + \omega^{-1} \Delta_{\mathcal{M}^\omega} \phi_k$$

Hippocampl mapping [Shi-Lai-Toga'14]

- Two hippocampal surfaces
  - Use 30 eigenfunctions in constructing the embedding space
  - Start with constant weights
- The weight of the source mesh are updated iteratively to compensate for the non isometric differences



Resulting conf. deform.



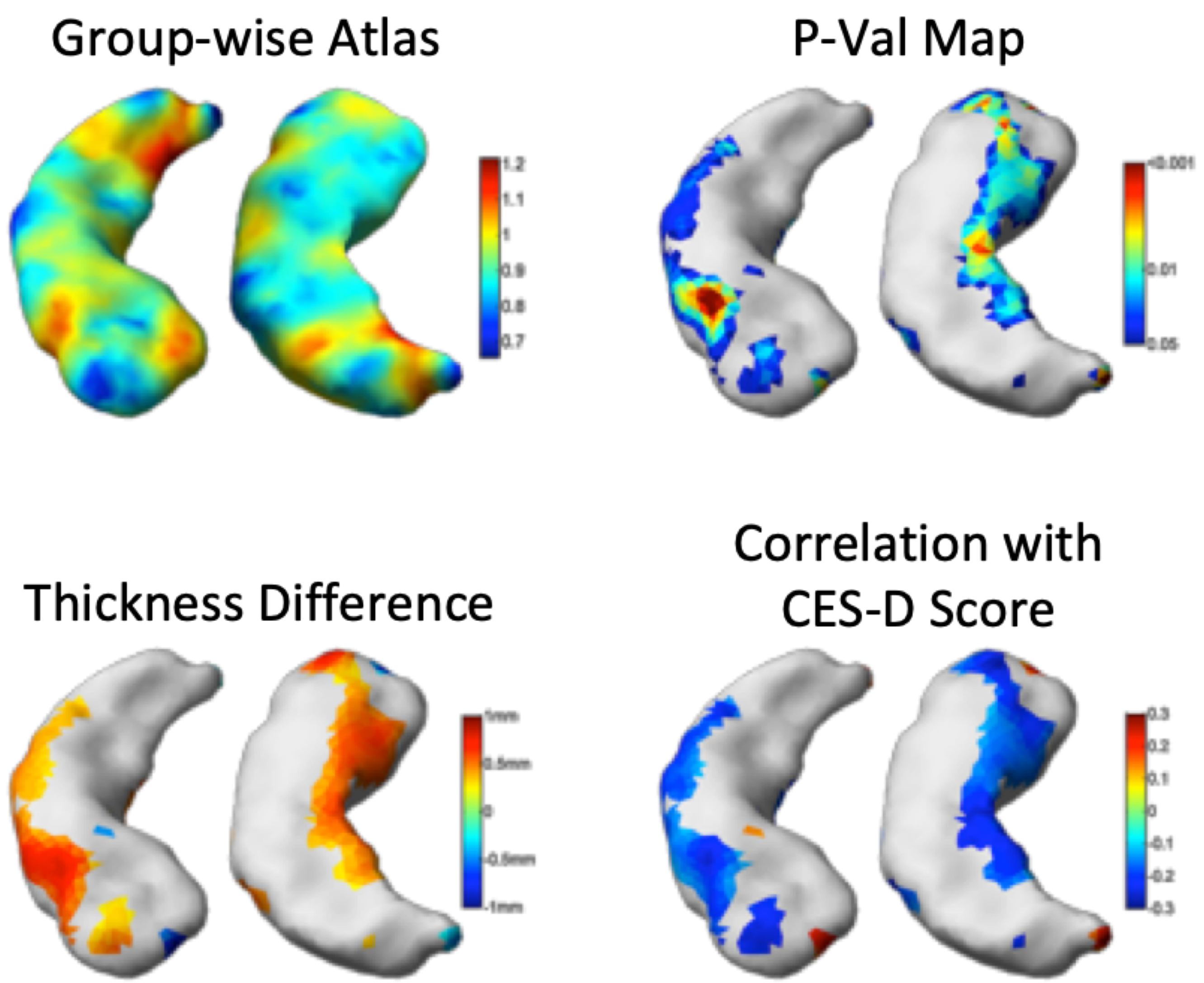
4, 5, 6, 7-th eigenfunctions after conf. deform

Before: Large distortion

After: high quality map

Clinical application of hippocampus mapping

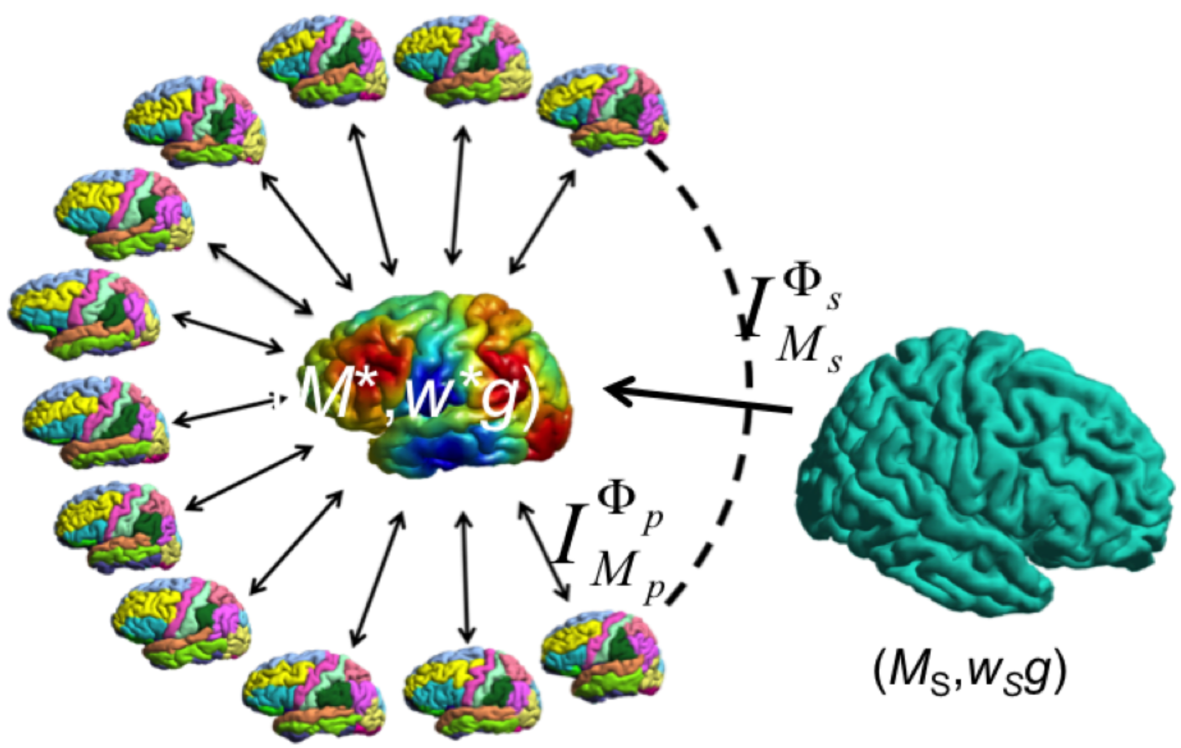
- Population study: hippocampal atrophy in multiple sclerosis (MS) patients with depression
  - 109 female patient split into two groups with the CES-D scale: low depression (CES-D≤20) and high depression (CES-D>20)
  - Statistically significant group differences were localized on hippocampus (P=0.019)
  - Correlates well with clinical measure of depression

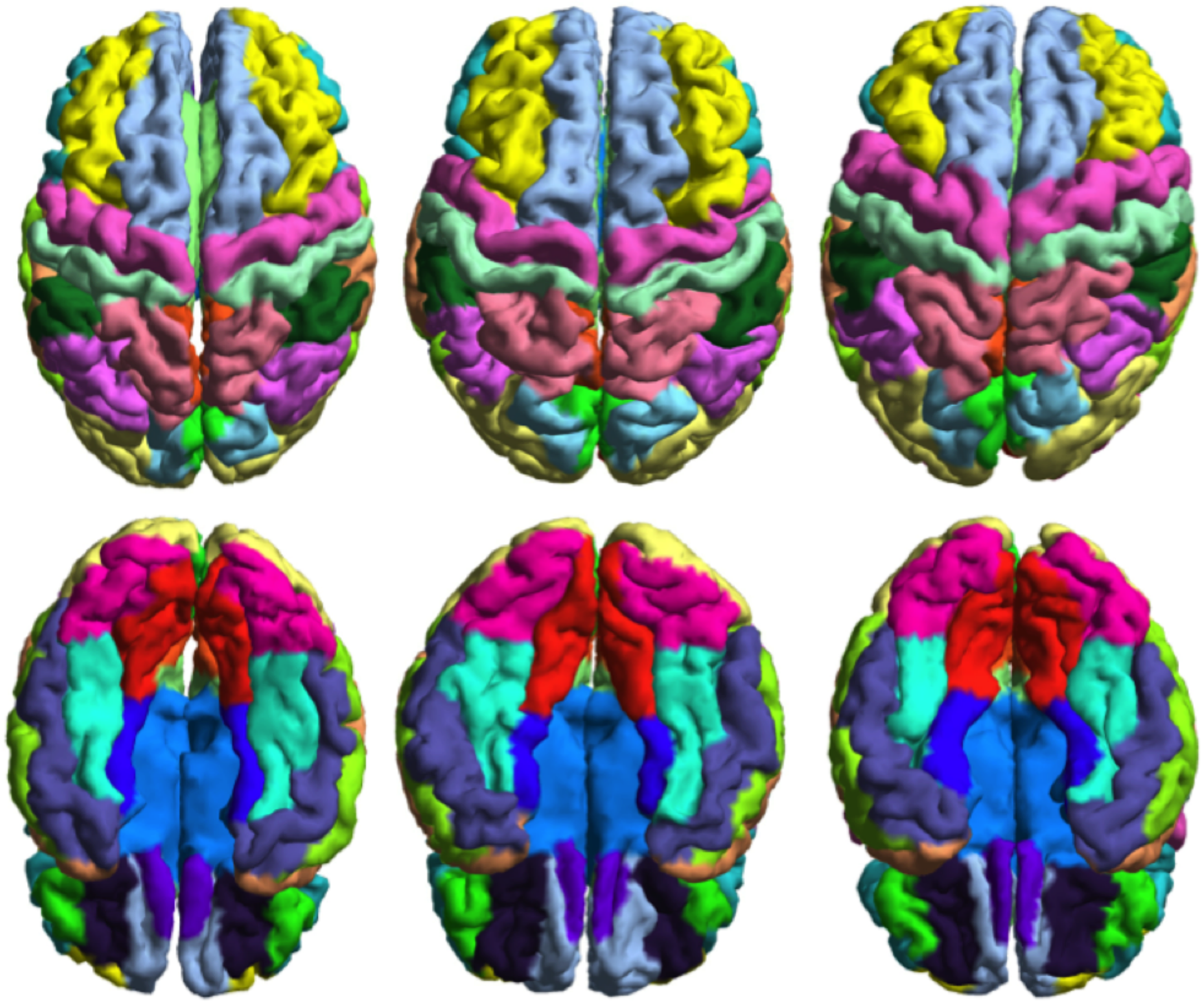


Cortical surface mapping [Shi-Lai-Toga'14]

Given a set of annotated surfaces $\mathcal{M}_1, \cdots, \mathcal{M}_k$, estimate a group-wise atlas $\mathcal{M}^\omega = (\mathcal{M}, \omega^* g)$ for cortical label fusion

$$\arg\min_{\omega} \sum_{i=1}^{k} D(\mathcal{M}_i, \mathcal{M}^\omega) + \xi \int_{\mathcal{M}} \|\nabla_{\mathcal{M}} \omega\|^2$$





Adaptively find a basis ϕ_i and conformation deformation on the target manifold

$$(T^*, w^*) = \arg\min_{T,w} \sum_{i=1}^{N} \int_{\mathcal{M}_1} \|\phi_i - \psi_i \circ T\|^2 \mathrm{d}\mathcal{M}_1 + \frac{1}{2} \sum_{i=1}^{N} \int_{\mathcal{M}_1} \|\nabla_{\mathcal{M}_2} \psi_i\|^2 \mathrm{d}\mathcal{M}_2,$$

$$\text{s.t.} \quad \int_{\mathcal{M}_2} \psi_i \psi_j \; w^2 \mathrm{d}\mathcal{M}_2 = \delta_{ij}$$

Functional Map

A smooth bijection $F : \mathcal{M}_1 \to \mathcal{M}_2$ induces a linear transformation

$$F_T : \mathcal{F}(\mathcal{M}_1, \mathbb{R}) \to \mathcal{F}(\mathcal{M}_2, \mathbb{R}), \quad f \mapsto f \circ F^{-1}$$

$$F_T(f) = \sum_i F_T(f_i \phi_i) = \sum_i f_i \sum_j c_{ij} \psi_i = \sum_i g_i \psi_i$$

The coefficient matrix c_{ij} store information of F [Ovsjanikov et al'12]

LB eigenbasis pursuit via conformal deformation

Given feature functions, adaptively find ψ_i and w.

$$(w^*, \Psi^*) = \arg\min_{w,\Psi} \frac{r_1}{2}\|\langle F, \Phi\rangle_{g_1} - \langle G, \Psi\rangle_{w^2 g_2}\|_F^2 + \frac{r_2}{2}\sum_{i=1}^{N}\int_{M_2} \|\nabla_{\mathcal{M}_2}\psi_i\|^2 \mathrm{d}\mathcal{M}_2 + \frac{r_3}{2}\int_{\mathcal{M}_2} ||\nabla_{\mathcal{M}_2} w||^2 \mathrm{d}\mathcal{M}_2,$$

$$\text{s.t.} \quad \int_{\mathcal{M}_2} \psi_i\psi_j \; w\mathrm{d}\mathcal{M}_2 = \delta_{ij} \quad \text{and } Area(\mathcal{M}_1)_{g_1} = Area(\mathcal{M}_2)_{w^2 g_2} \tag{1}$$

- Functional maps: A smooth bijection $F : \mathcal{M}_1 \to \mathcal{M}_2$ induces a linear transformation

  $$F_T : \mathcal{F}(\mathcal{M}_1, \mathbb{R}) \to \mathcal{F}(\mathcal{M}_2, \mathbb{R}), \quad f \mapsto f \circ F^{-1}, \quad F_T(f) = \sum_i F_T(f_i\phi_i) = \sum_i f_i \sum_j c_{ij}\psi_i = \sum_i g_i\psi_i$$

  The coefficient matrix c_{ij} store information of F [Ovsjanikov et al'12]

- Deformed eigensystem:Let $\{\phi_n^{w^2}, \lambda_n^{w^2}\}_n$ be a LB eigensystem of a conformally deformed surface $(\mathcal{M}, w^2 g_{\mathcal{M}})$, then $\{\phi_n^{w^2}, \lambda_n^{w^2}\}_n$ is equivalent to the following weighted LB eigensystem on $(\mathcal{M}, g_{\mathcal{M}})$:

  $$\sum_i \int_{\mathcal{M}} ||\nabla_{\mathcal{M}}\phi_i||^2 d\mathcal{M} \quad \text{s.t.} \int_M \phi_i\phi_j w^2 \mathrm{d}vol_{g_{\mathcal{M}}} = \delta_{ij}$$

- Smoothness of the conformal deformation

Proximal alternating method

$\mathcal{S} = \{\bar{\Psi} \in \mathbb{R}^{n\times N} \mid \bar{\Psi}^T\bar{\Psi} = \mathbf{I}_n\}$ and $\mathcal{W} = \{w \in \mathbb{R}^n \mid \sum_{i=1}^n w_i^2 \mathbf{M}_{ii}^2 = A\}$
and define indicator functions:

$$\delta_{\mathcal{S}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{S} \\ +\infty, & \text{otherwise} \end{cases}, \qquad \delta_{\mathcal{W}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{W} \\ +\infty, & \text{otherwise} \end{cases}.$$

We write the equivalent form of the optimization problem as:

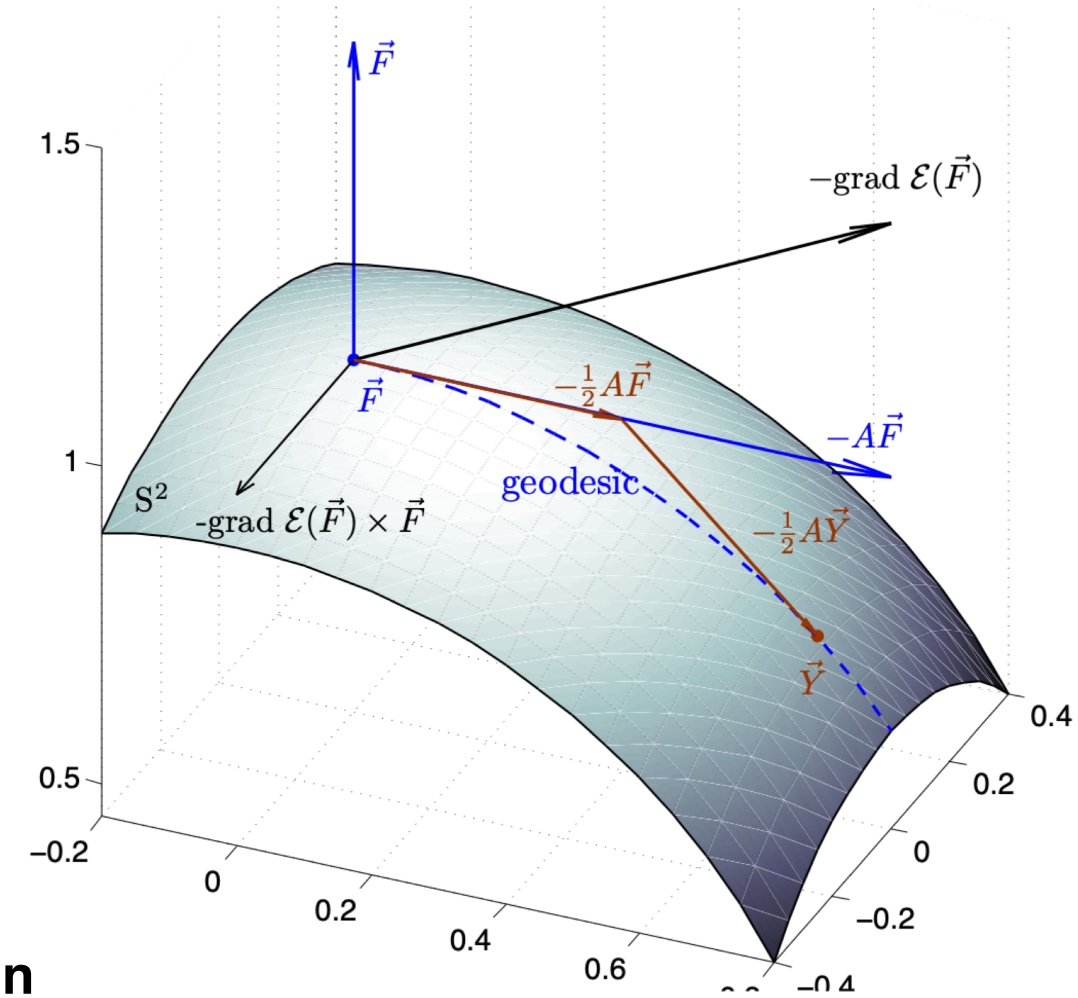
$$(w^*, \bar{\Psi}^*) = \arg\min_{w,\bar{\Psi}} \mathcal{E}(w, \bar{\Psi}) + \delta_{\mathcal{S}}(\bar{\Psi}) + \delta_{\mathcal{W}}(w)$$

Then the proximal alternating method will be:

$$\begin{cases} \bar{\Psi}^{j+1} = \arg\min_{\bar{\Psi}} \mathcal{E}(w^j, \bar{\Psi}) + \frac{1}{2\eta}||\bar{\Psi} - \bar{\Psi}^j||^2, \quad \text{s.t.} \quad \bar{\Psi}^T\bar{\Psi} = \mathbf{I}_n. \qquad \text{Curilinear search} \\ w^{j+1} = \arg\min_{w} \mathcal{E}(w, \bar{\Psi}^{j+1}) + \frac{1}{2\eta}||w - w^j||^2, \quad \text{s.t.} \quad \sum_{i=1}^n w_i^2 \mathbf{M}_{ii}^2 = A. \qquad \text{ADMM} \end{cases}$$

Curvilinear search with BB step size [Wen-Yin'13]

$$\min_X \mathcal{F}(X) \quad \text{s.t.} \quad X^T X = I$$



- $G = \dfrac{\partial \mathcal{F}(X)}{\partial X_{ij}}$
- $\nabla^c_{\mathcal{M}} \mathcal{F}(X) = (GX^T - XG^T)X = AX$
- $X^{new} = X - \tau A(\frac{X+X^{new}}{2})$

$$\Longrightarrow X^{new} = \left(I + \tfrac{\tau}{2}A\right)^{-1}\left(1 - \tfrac{\tau}{2}A\right)X$$

Cayley transformation

Remark (Wen-Yin'13). *1. τ can be chosen as the well-know Barzilai-Borwein (BB) step size to accelerate the gradient descent method.*

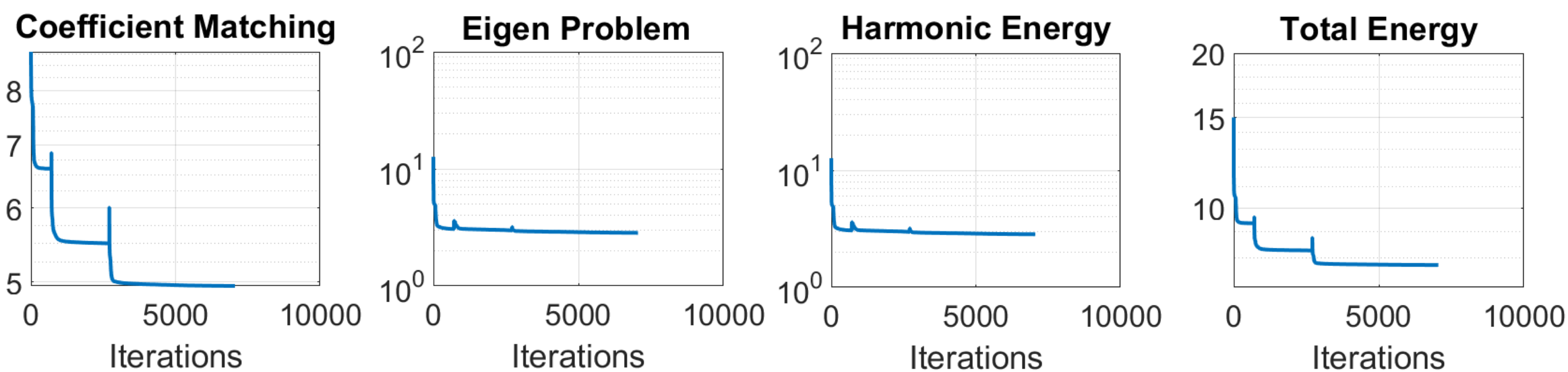
2. A fast way of computing $(I+\frac{\tau}{2}A)^{-1}$ can be obtained using the Sherman-Morrison-Woodbury (SMW) formula.

3. With certain condition, the sequence $\{X^k\}$ generated by the algorithm satisfies $\lim_{k\to\infty} \|\nabla\mathcal{F}(X^k)\|_F = 0$

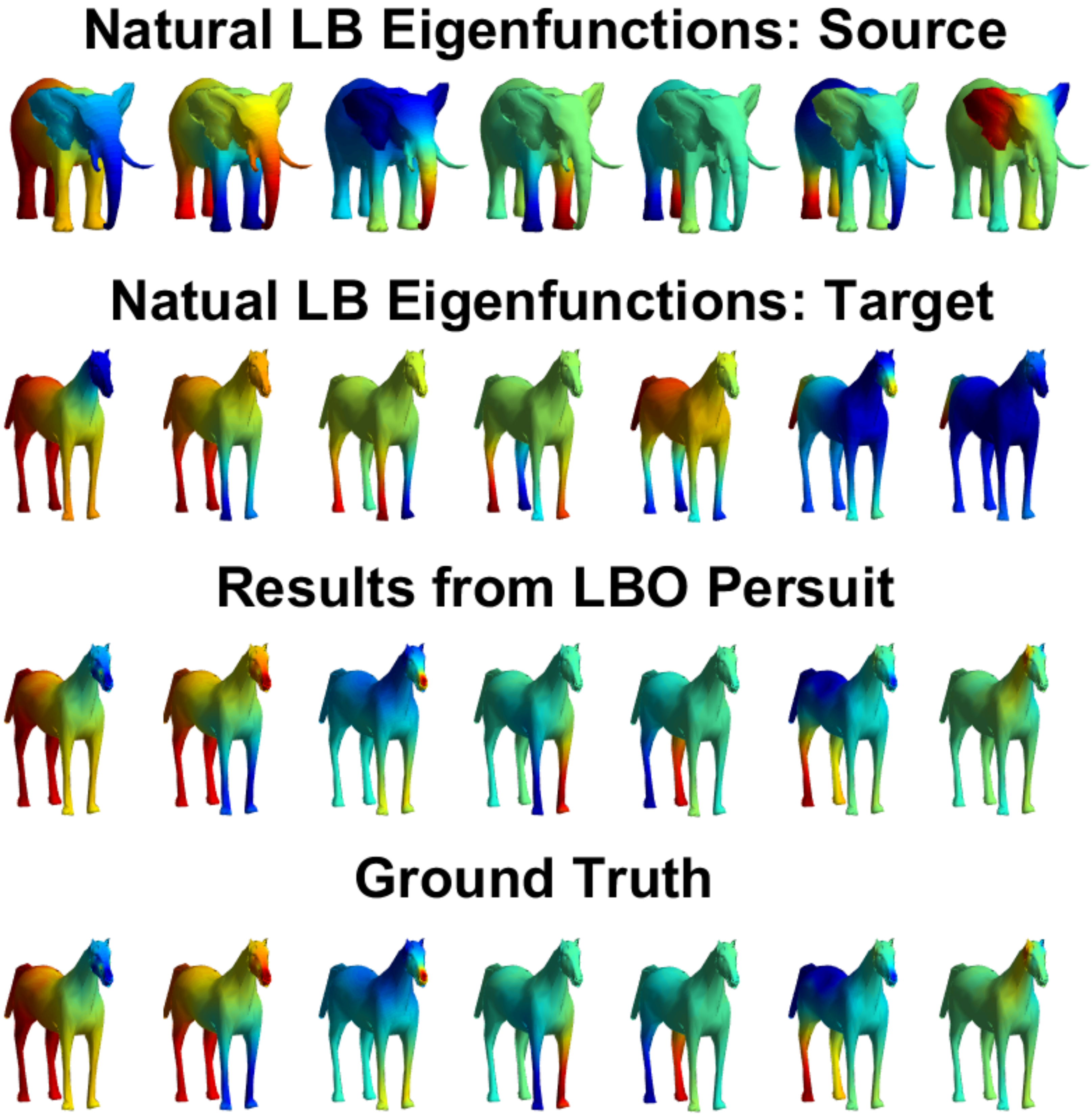
Convergence

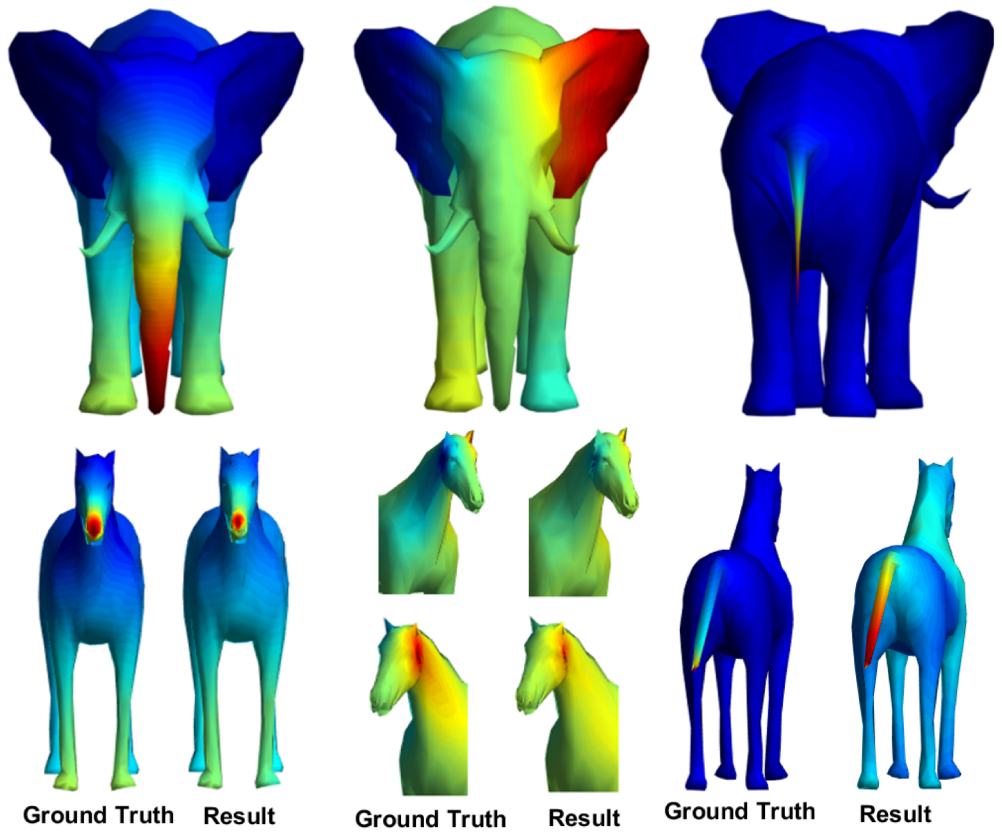
Theorem 1 (local convergence). *Let* $\{w^j, \bar{\Psi}^j\}$ *be the sequence produced by the above algorithm, then:*

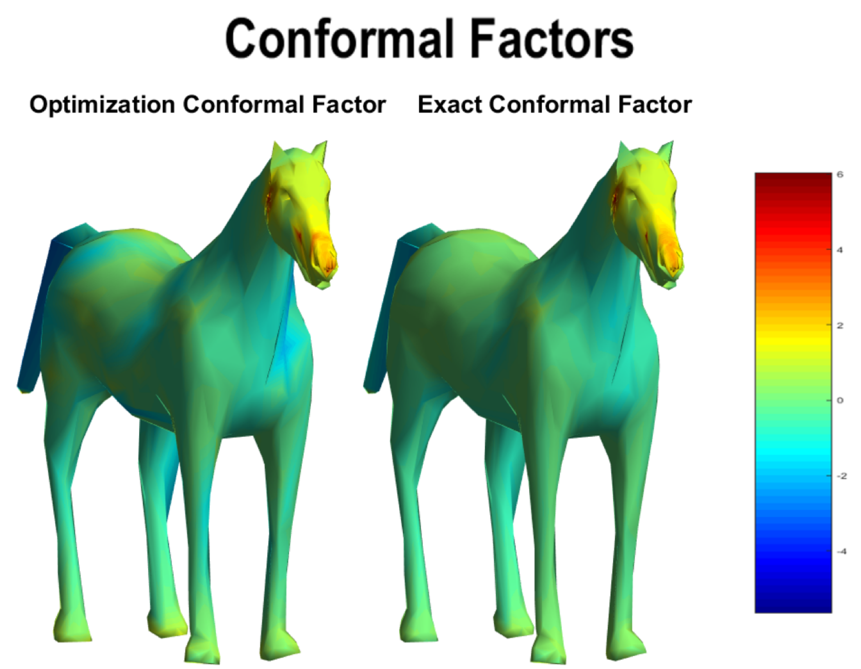
- $\mathcal{E}(w^{j+1}, \bar{\Psi}^{j+1}) + \frac{1}{2\eta}||\bar{\Psi}^{j+1} - \bar{\Psi}^j||^2 + \frac{1}{2\eta}||w^{j+1} - w^j||^2 \leq \mathcal{E}(w^j, \bar{\Psi}^j),\ \forall j \geq 0.$
- $\sum_{j=1}^{\infty}(\|w^j - w^{j-1}\|^2 + \|\bar{\Psi}^j - \bar{\Psi}^{j-1}\|^2) < \infty.$
- $\{w^j, \bar{\Psi}^j\}$ *converges to a critical point of* $\mathcal{E}(w, \bar{\Psi})$.



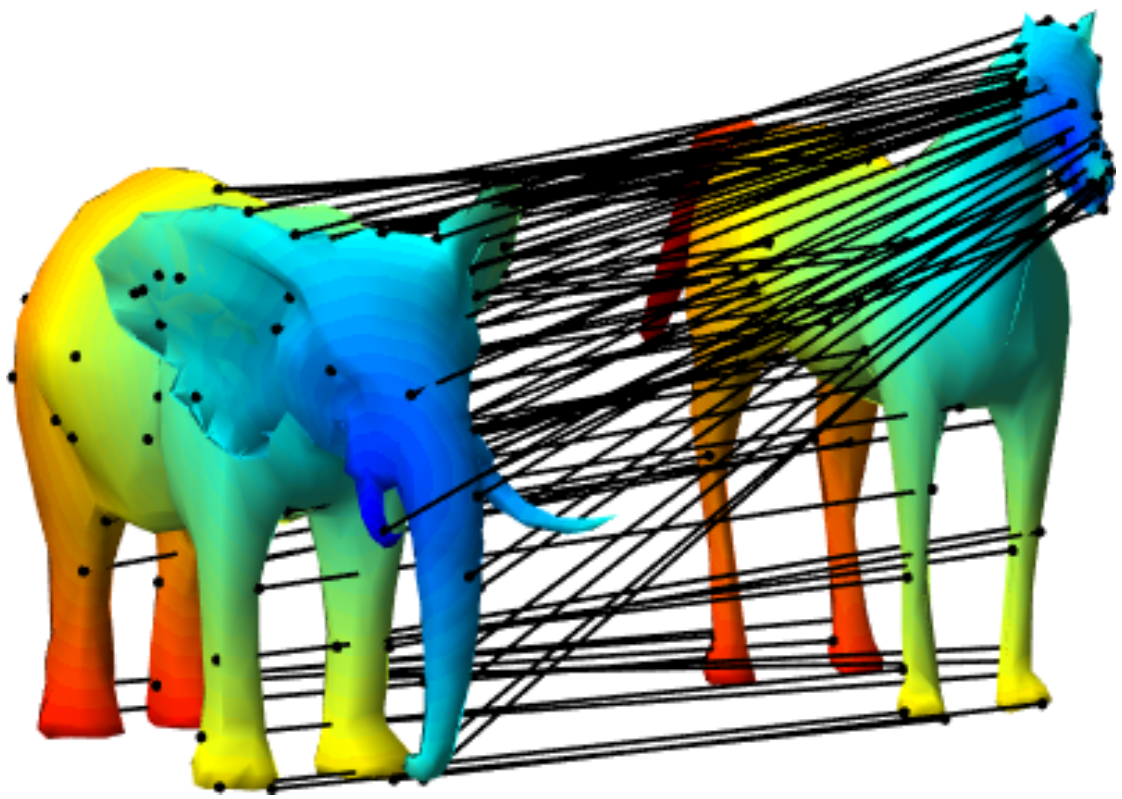
Results of LB basis pursuit and conformal factor

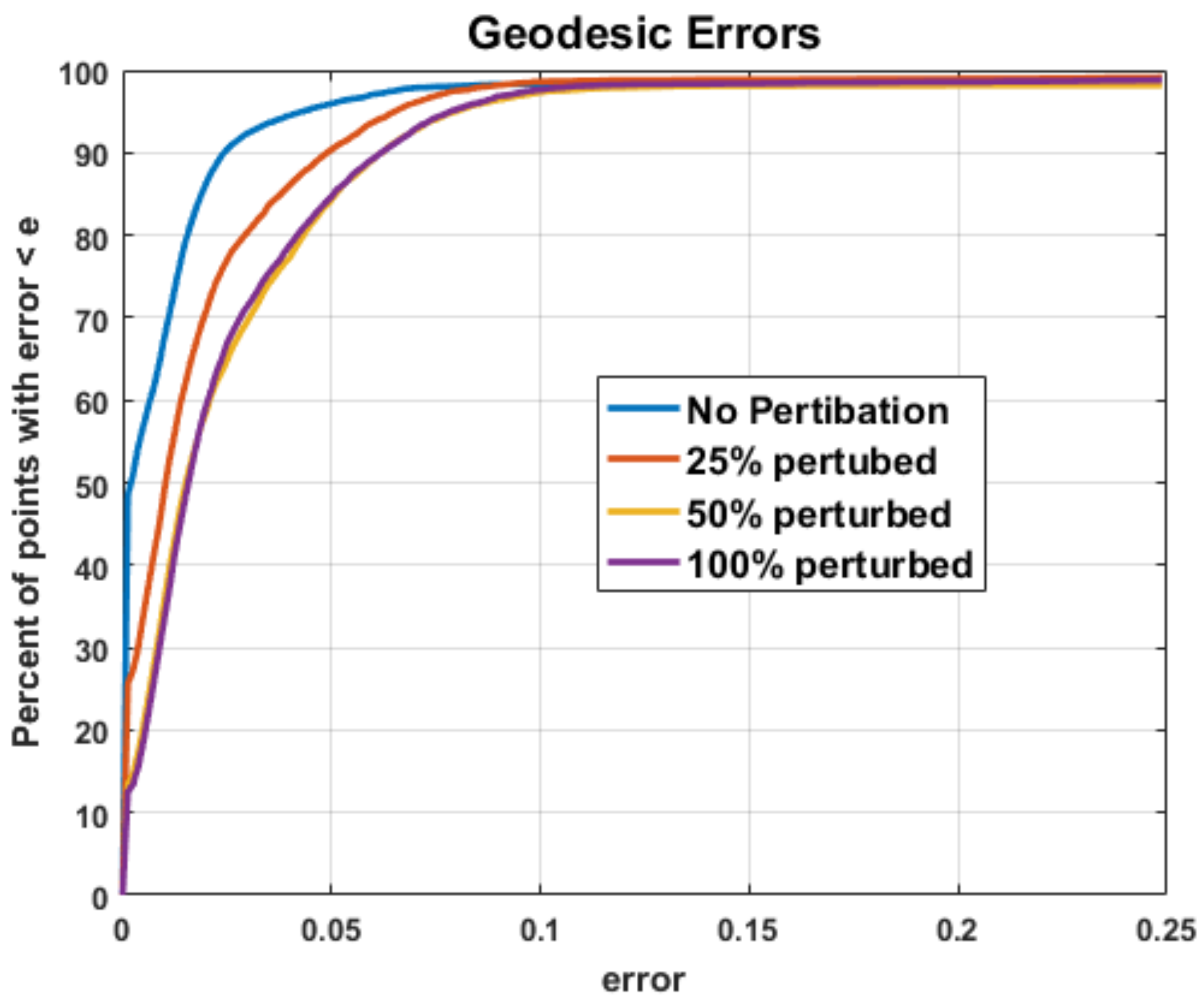




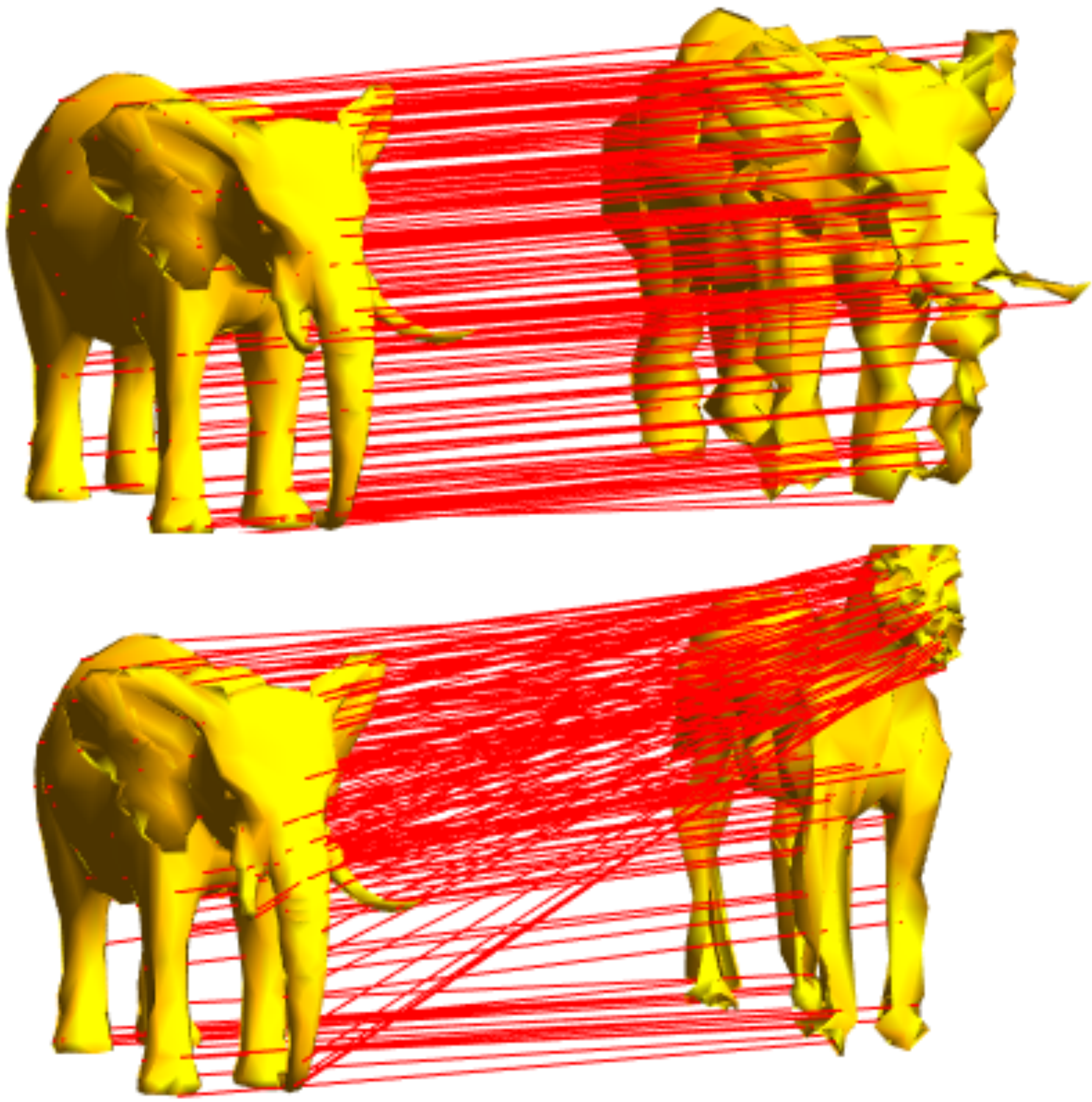


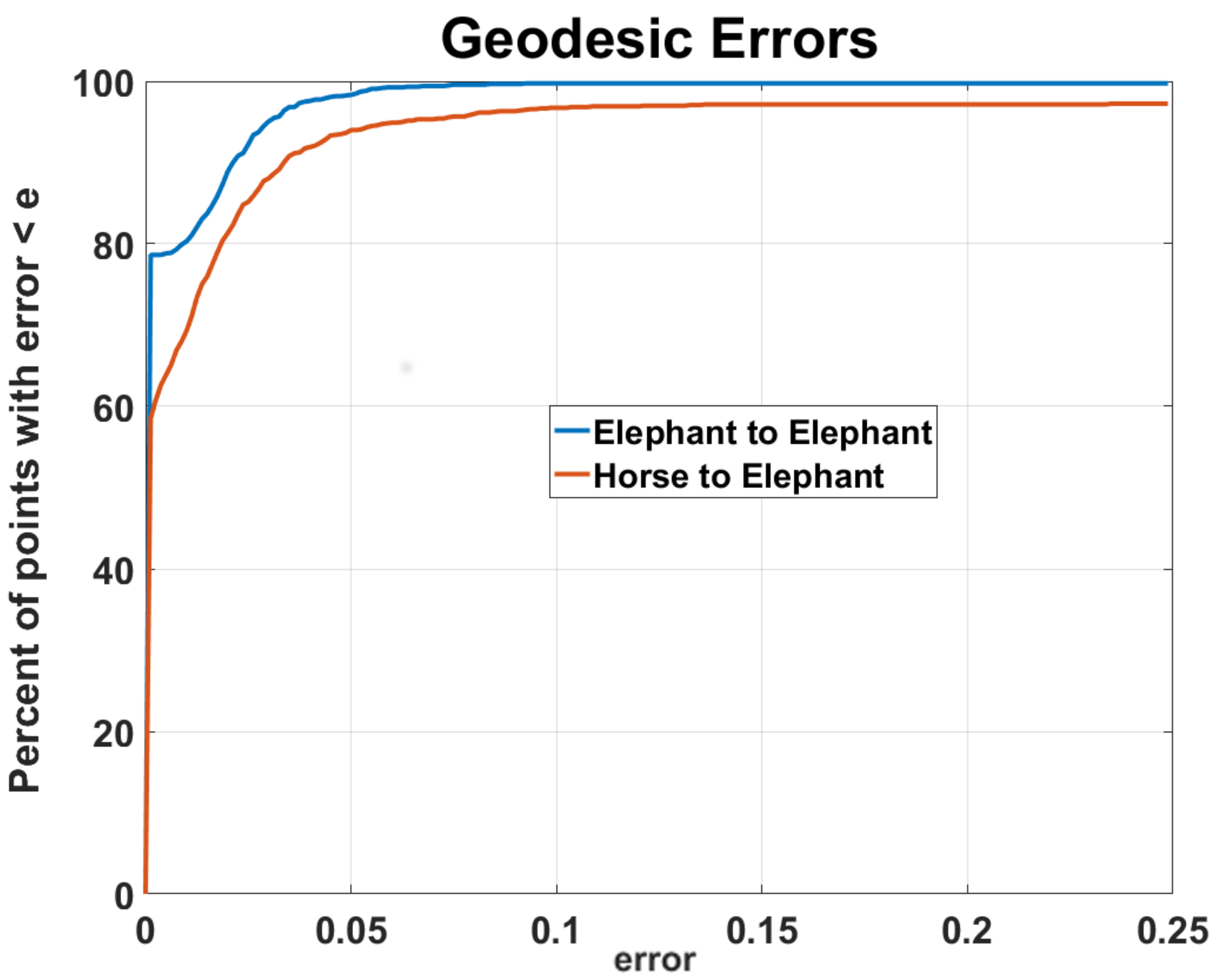
Accuracy and Robustness





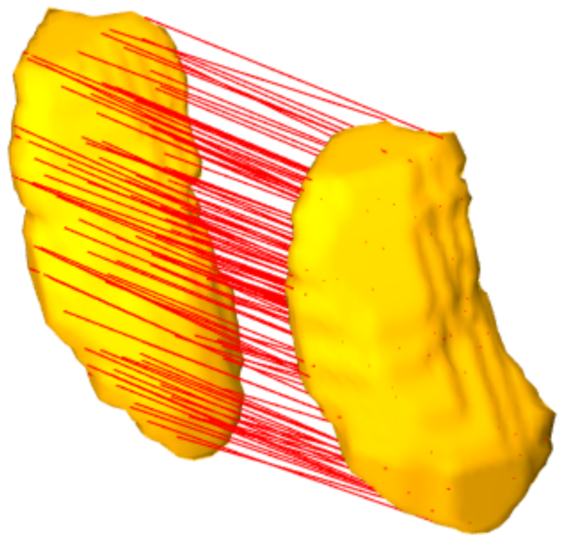
Noisy case

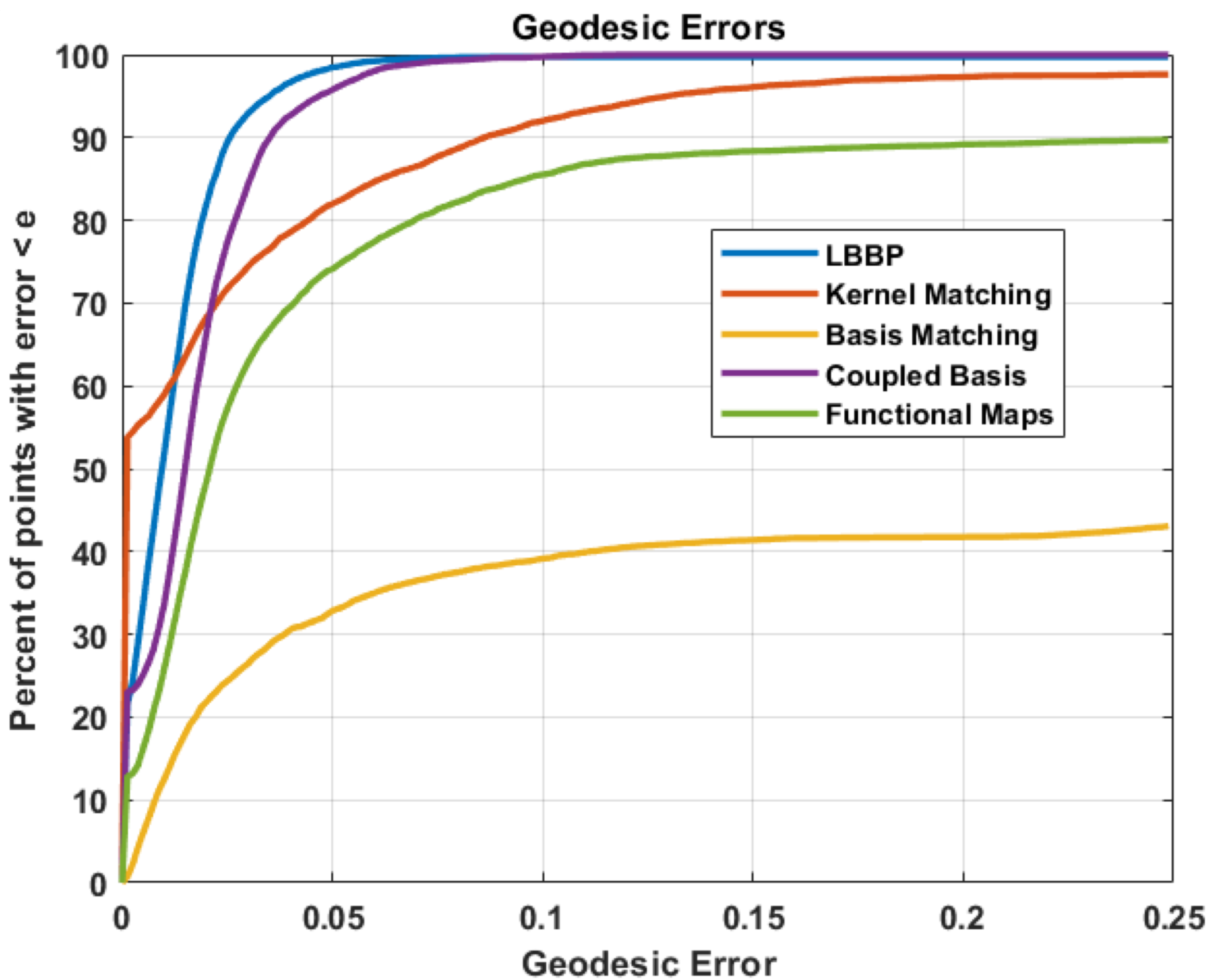


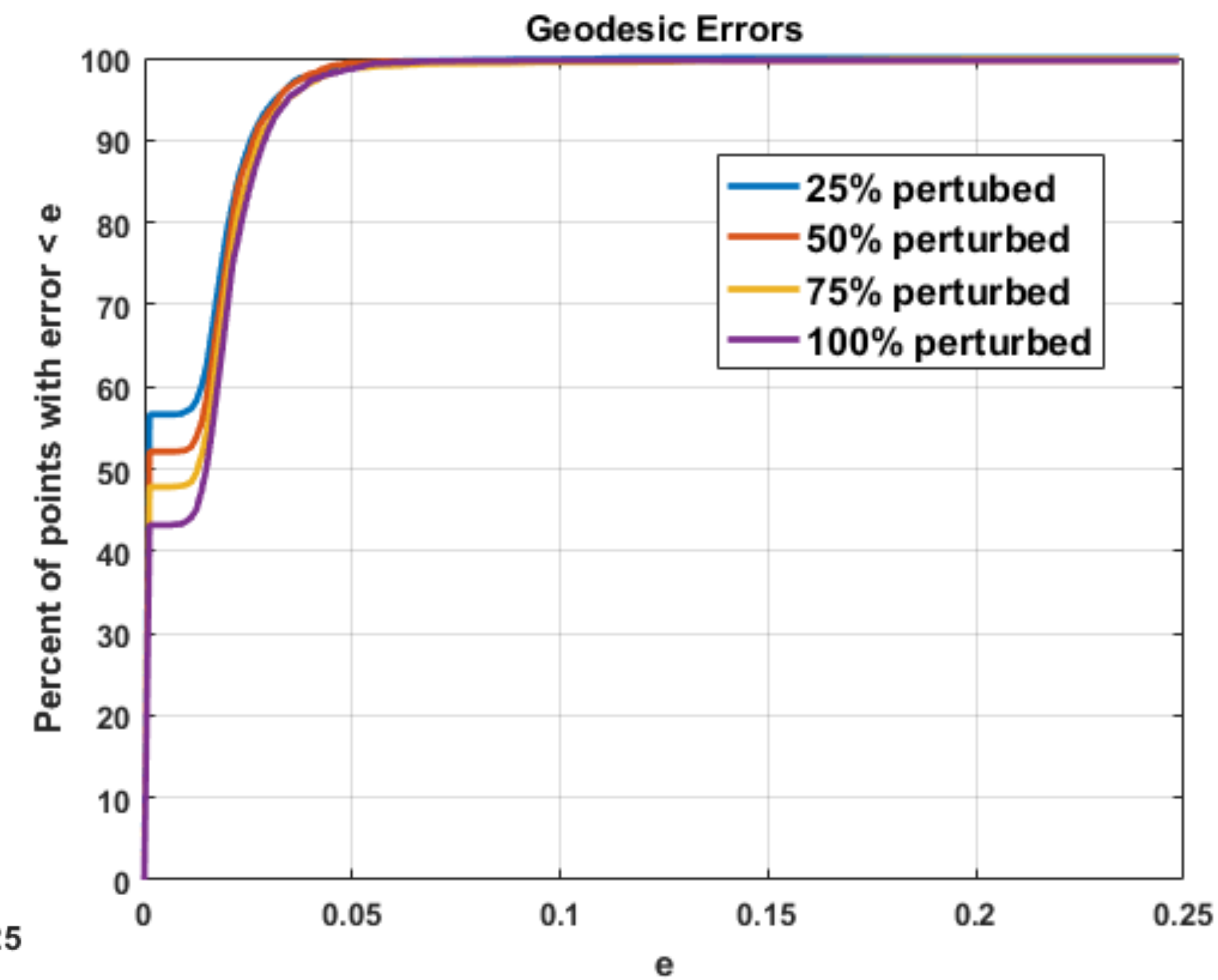


Geodesic Errors
Percent of points with error < e
error
0
20
40
60
80
100
0
0.05
0.1
0.15
0.2
0.25
Elephant to Elephant
Horse to Elephant

Results on FAUST and human brain mapping







Combining modeling and learning [MS A1-1-1, 18/07, 14:30]

- Traditional modeling based on understanding prior knowledge of the objects, not relying too much on large amount of available data sets.
- Learning methods require large amount of training data sets, however, it is hard to achieve reasonable performance of our problems as the data has representation ambiguities as a high dimensional isometric group.

Can we combine advantages of both methods in our problem?

- No need to design and solve a specific variational PDE model, instead, use **data driving** methods.
  - **Convolution** is certain discretization of differenti

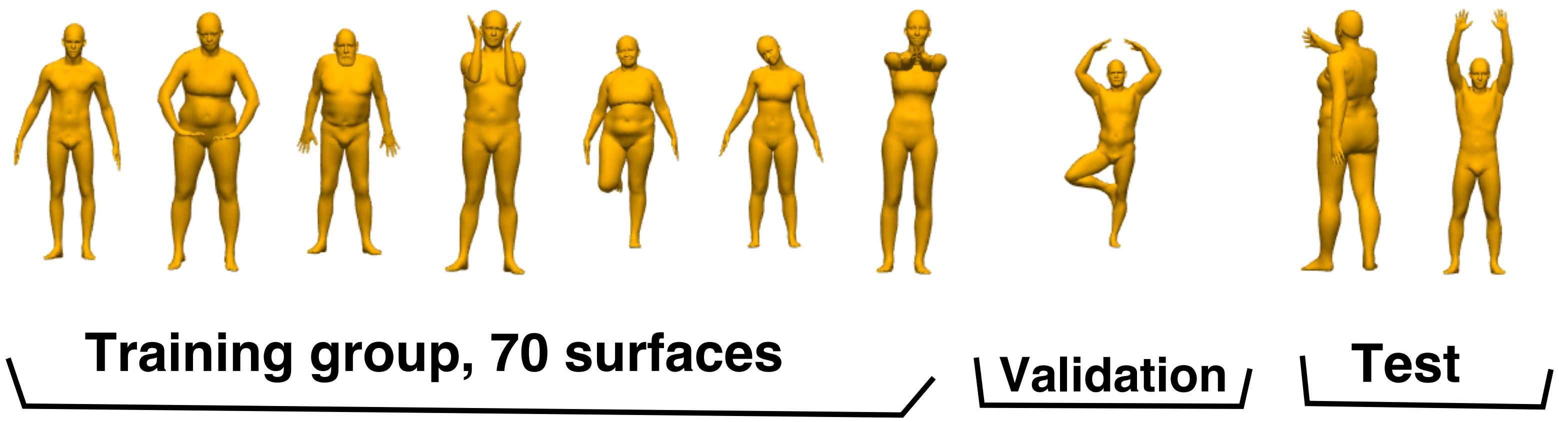
$$\min_u \int |\nabla_{\mathcal{M}} u|^2 \quad \Longrightarrow \quad \frac{\partial u}{\partial t} = \Delta_{\mathcal{M}} u$$

  - Use certain modeling method to **overcome rep**

Parallel transport convolution, translation, dilation, rotation

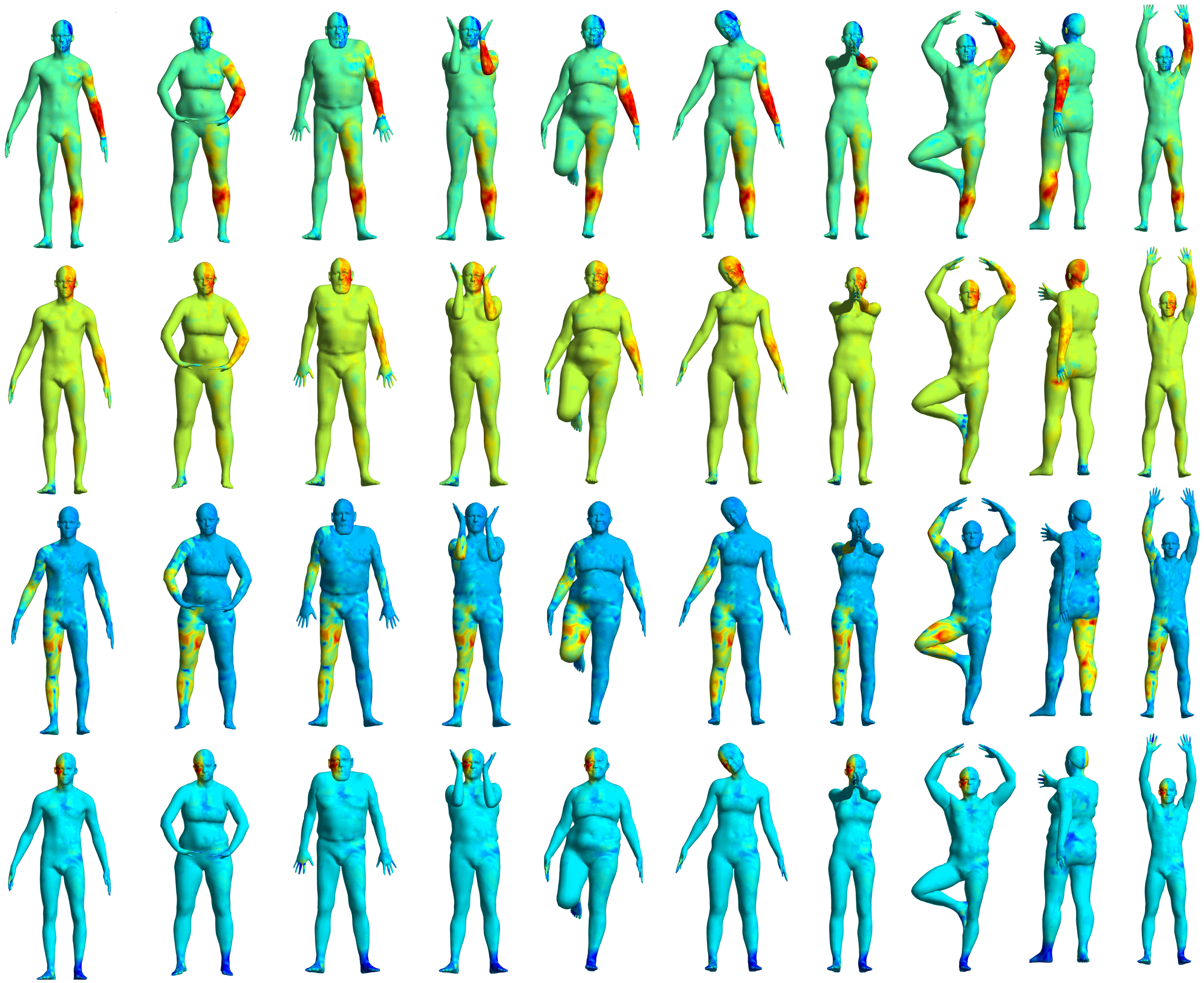
Original Kernel Translation Dilation Rotation Mixed

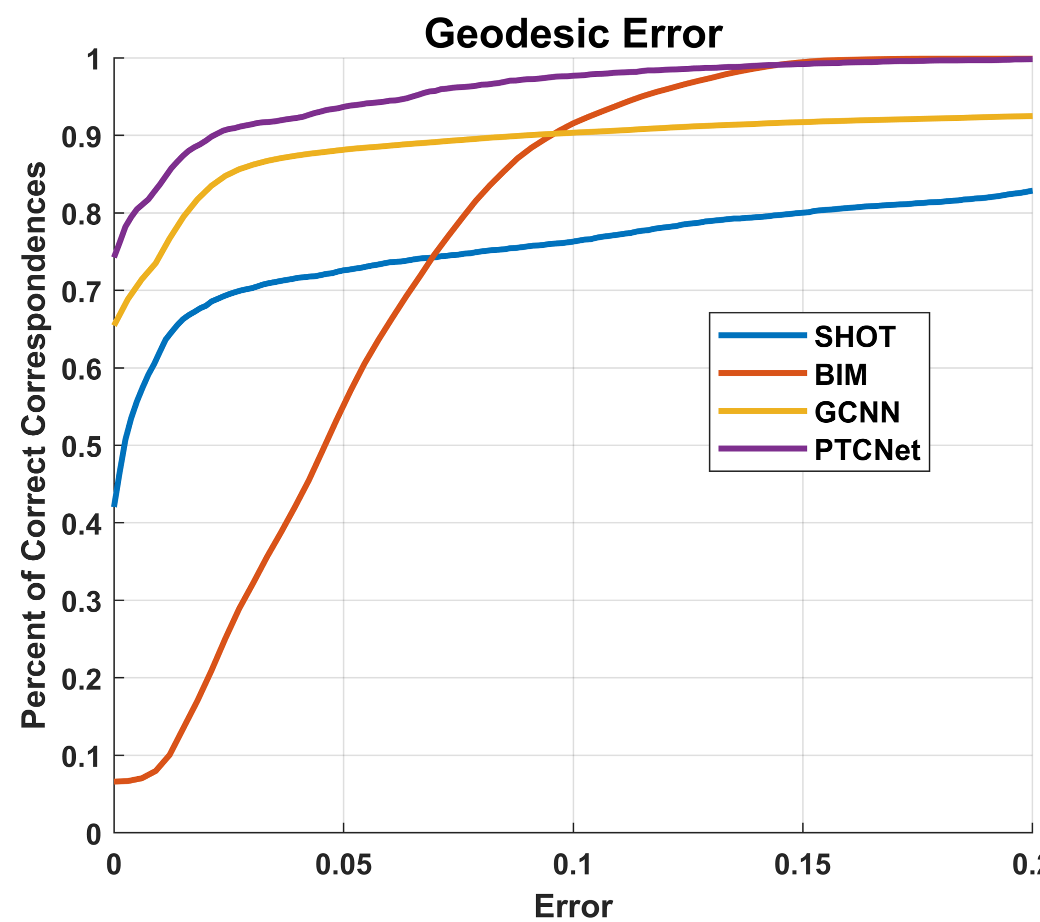
Manifold registration using PTCNN



FAUST data set

- 100 surfaces of 10 persons in 10 poses;
- 7 (70 surfaces) for training
- 1 (10 surfaces) for validation
- 2 (20 surfaces) for testing





Summary

- We combine the spectral geometry and optimal transport to conduct shape matching.
- We use conformal deformation to characterize the shape space.
- We propose variational PDE methods to compute deformation and LB eigenbasis.

Supported in part by NSF DMS-1522645 and an NSF CAREER Award (DMS-1752934)

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