Multiscale Monogenic Image Representations Using Poisson Kernels

Minisymposium on *Recent Advances in Multiscale Transforms for Image Analysis*
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Motivation

1D Analytic Signal

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This Minisymposium Focuses on:

- Multiscale image transforms beyond wavelets
- Deeper connections between harmonic analysis and image analysis
- Various methods to decompose an image into “predictable” local segments and their residuals that allow efficient and sparse image approximation

Saito:	Exploration of multiscale “monogenic” image representations
Morita:	Image interpolation using PCA based on gradient information and boundary data
Ashizawa:	An improved coding scheme using multi-neighbor predictors and residual orthogonal transformations
Fujinoki:	A new 2D discrete wavelet transform that can handle 12 multiscale directions

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Brian Knight (UCD)

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Motivation

- The analytic signal is a ubiquitous tool in 1D signal processing and is used to analyze signals with *time-varying frequencies* as a natural way to compute the instantaneous phase and amplitude.
- The core idea is, given a real-valued signal, $f : \mathbb{R} \to \mathbb{R}$, i.e. some measured data, there is a 'natural' choice for an imaginary component, $\tilde{f}$, to pair it with, which 'completes' the data in some sense, perhaps in allowing f to take the form

$$f(t) = a(t)\cos(\phi(t)).$$

- Gabor (1946) proposed that the natural choice is the *Hilbert transform* of f.

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1D Analytic Signal

- Vakman (1972) showed that the *Hilbert transform* is a unique choice for $\tilde{f}$ under certain assumptions, and called $u(t) = f(t) + \mathrm{i}\tilde{f}(t) = f(t) + \mathrm{i}\mathscr{H}f(t)$ the *analytic signal*. The assumptions are as follows:
    - $\tilde{f}$ must be derived from f
    - continuity of amplitude: a small change in f gives a small change in $a(t)$
    - phase independence of scale: scaling f does not alter the phase function of the resulting complex signal
    - harmonic correspondence: if $f(t) = a_0 \cos(\omega_0 t + \phi_0)$, then $a(t) = a_0$ and $\phi(t) = \omega_0 t + \phi_0$, i.e., $\tilde{f}(t) = a_0 \sin(\omega_0 t + \phi_0)$.
- We will focus on *periodic* signals $f : \mathbb{T} \to \mathbb{R}$ in which case

$$\mathscr{H}f(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau) \cot\left(\frac{t-\tau}{2}\right) \mathrm{d}\tau.$$

- The Hilbert kernel $h(t) := \frac{1}{2\pi} \cot\left(\frac{t}{2}\right)$ has Fourier coefficients $\hat{h}_n = -\mathrm{i}\,\mathrm{sign}(n)$, whence $u(t) = \hat{f}_0 + 2\sum_{n \geq 1} \hat{f}_n \mathrm{e}^{\mathrm{i}nt}$.

IAP Representation

- Even with the uniqueness of the analytic signal, its instantaneous amplitude and phase (IAP) representation $u(t) = a(t)\mathrm{e}^{\mathrm{i}\phi(t)}$ is not unique [Cohen-Loughlin-Vakman (1999)]. Either we require
  - $a(t) \geq 0$, in which case $\phi(t)$ *may be discontinuous*; or
  - $\phi(t)$ *to be continuous*, in which case $a(t)$ *may be negative*.
- A good example of this is given in their paper:

$$f(t) = \frac{1}{2}\cos(\omega_a t) + \frac{1}{2}\cos(\omega_b t) = \cos(\omega_1 t)\cos(\omega_2 t),$$

  where $\omega_a = \omega_2 + \omega_1, \omega_b = \omega_2 - \omega_1$.

- With $\omega_2 > \omega_1 \geq 0$, the analytic signal is given by $u(t) = \cos(\omega_1 t)\mathrm{e}^{\mathrm{i}\omega_2 t}$ (a nice instance of Bedrosian's theorem), but a usual procedure for computing amplitude and phase gives the IAP representation

$$u(t) = |\cos(\omega_1 t)|\mathrm{e}^{\mathrm{i}\omega_2 t},$$

  which implies $\mathrm{Re}\, u(t) = |\cos(\omega_1 t)|\cos(\omega_2 t) \neq f(t) = \cos(\omega_1 t)\cos(\omega_2 t)$.

- We will look at a similar example in the case of the 2D analytic signal!

1D Analytic Signal

- Given $f \in L^2(\mathbb{T})$, since u has only positive Fourier coefficients, we have that $u \in H^2(\mathbb{T})$, so u can be viewed as *the boundary value of an analytic function*,

  $$U(z) = F(z) + \mathrm{i}\tilde{F}(z), \qquad z \in \mathbb{D},$$

  where

  $$U(z) = U(r\mathrm{e}^{\mathrm{i}t}) = \int_{-\pi}^{\pi} (P_r(\tau) + \mathrm{i}Q_r(\tau)) f(t-\tau)\,\mathrm{d}\tau$$

  with $P_r(t) := \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(t)+r^2}$, $Q_r(t) := \frac{1}{2\pi}\frac{2r\sin(t)}{1-2r\cos(t)+r^2}$, are the *Poisson kernel* and the *conjugate Poisson kernel* for $\mathbb{D}$, respectively.

The Blaschke Product

- By viewing our real-valued signal f on $\mathbb{T}$ as the real part of the *boundary value* of an analytic function U on $\mathbb{D}$, and using the *Blaschke product*-based factorization method, we can avoid the ambiguity issues in the previous IAP representation [Picinbono (1997); Nahon (2000); Qian (2009); Qian-Wang (2011); Coifman-Steinerberger (2017)].
- Supposing that $U \in H^p(\mathbb{D})$ for $\exists p > 0$. Then we can *factorize* U as

  $$U(z) = B(z)G(z),$$

  where $B(z)$ is a *Blaschke product* containing all zeros of U in $\mathbb{D}$, and $G(z) \neq 0$ for $z \in \mathbb{D}$.
- If $U(z)$ has a finite number of zeros in $\mathbb{D}$, $B(z)$ takes the following form:

  $$B(z) = z^N \prod_{k=1}^{M} \left(\frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \frac{\overline{\alpha}_k}{|\alpha_k|} \right).$$

- On the boundary, this yields

  $$u(t) = b(t)g(t), \quad t \in \mathbb{T},$$

  where $b(t) = B(\mathrm{e}^{\mathrm{i}t})$ is called the *phase signal*.

The Blaschke Product

- $|b(t)| = |B(\mathrm{e}^{\mathrm{i}t})| = 1$, and $g(t) \neq 0$, which motivates our interpretation of *$b(t) =$ instantaneous phase* and *$g(t) =$ instantaneous amplitude*.
- In fact, if we write $B(\mathrm{e}^{it}) = B(1)\mathrm{e}^{\mathrm{i}(\phi_b(t)-\phi_b(0))}$, it can be shown that

  $$\phi_b'(t) = N + \sum_{k=1}^{M} \frac{1-|\alpha_k|^2}{|\mathrm{e}^{\mathrm{i}t}-\alpha_k|^2} > 0,$$

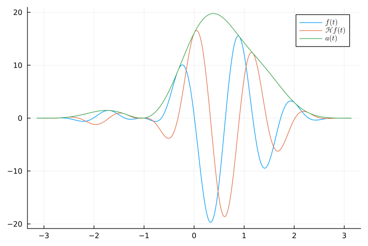
  so the phase $\phi_b(t)$ is non-decreasing and the instantaneous frequency is non-negative, which resolves the 1D phase unwrapping problem.
- Further, on the boundary we have $|u(t)| = |U(\mathrm{e}^{\mathrm{i}t})| = |G(\mathrm{e}^{\mathrm{i}t})| = |g(t)|$, so we take g to be a (complex-valued) instantaneous amplitude.

The Blaschke Product: An Example

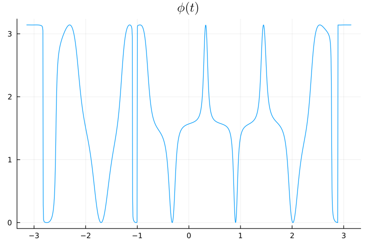
Consider

$$U(z) = (z+0.8)^5 (z-0.98\mathrm{e}^{\mathrm{i}\pi/3})^2 (z-0.5\mathrm{e}^{\mathrm{i}\pi/3})$$

The “standard” IAP representation of $f(t) = \mathrm{Re}(U(z))|_{\partial\mathbb{D}}$:



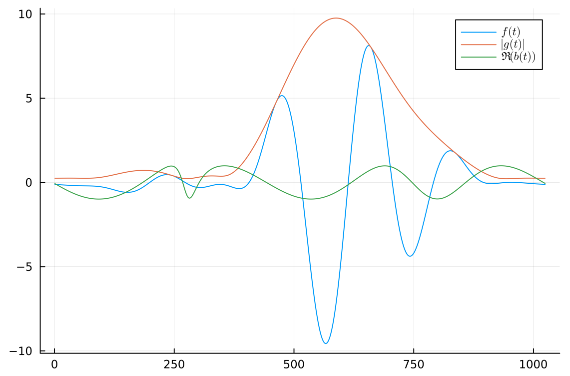
(a) blue: $f(t)$, red: $\mathscr{H}f(t)$, green: $a(t)$



(b) Discontinuous IAP phase $\phi(t)$

The Blaschke Product: An Example

$$U(z) = (z+0.8)^5 (z-0.98e^{i\pi/3})^2 (z-0.5e^{i\pi/3})$$

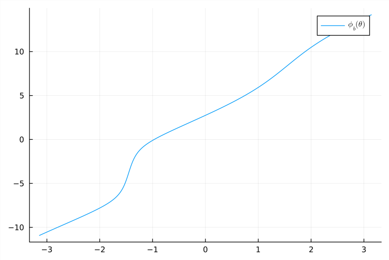


blue: $f(t)$, red: $g(t)$ (Blaschke amplitude), green: $\mathrm{Re}(b(t))$

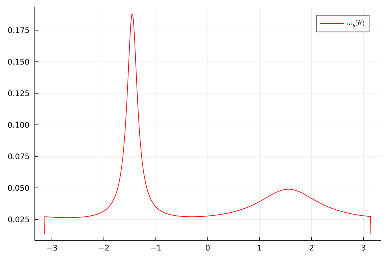
The Blaschke Product: An Example

$$U(z) = (z+0.8)^5 (z-0.98\mathrm{e}^{\mathrm{i}\pi/3})^2 (z-0.5\mathrm{e}^{\mathrm{i}\pi/3})$$

Using the factorization using the Blaschke product, we can get:



(a) Instantaneous phase $\phi_b(t)$



(b) Instantaneous frequency $\omega(t) = \phi_b'(t)$

Analytic Scale Space

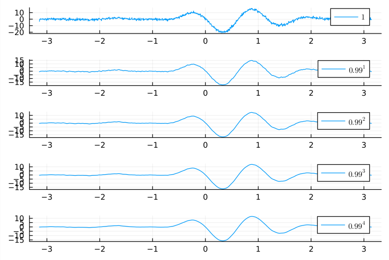
- Another insight in viewing our original analytic signal $u(t)$ as the *boundary value* of an analytic function in $\mathbb{D}$, i.e., $u(t) = U(\mathrm{e}^{\mathrm{i}t})$, is the development of the *analytic (or Poisson) scale space*.
- The Cauchy kernel $C_r := P_r + \mathrm{i}Q_r$ for each fixed $0 < r < 1$ acts as an interesting low-pass filter on f:

$$C_r * f(t) = \hat{f}_0 + 2\sum_{n\geq 0} \hat{f}_n r^n \mathrm{e}^{\mathrm{i}nt}.$$

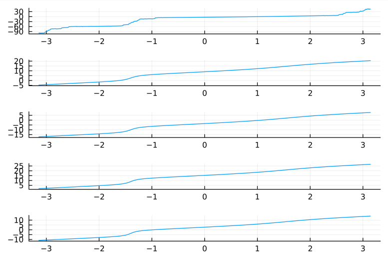
- It suppresses high frequency information exponentially as $r \downarrow 0$, and the resulting signal is always analytic.
- Felsberg and Sommer (2002) formalized this notion of scale space (in 2D) as it relates to the more standard Gaussian scale space.

Analytic Scale Space: An example

Let $f(t)$ be the real part of $U(\mathrm{e}^{\mathrm{i}t})$; $t \in [-\pi, \pi)$, and consider then the noisy signal $f(t) + \eta(t)$ where $\eta \sim N(0, 1)$.



(a) $P_r * f(t)$ for $r = 0.99^k$ for $k = 1, \ldots, 5$



(b) Corresponding phase $\phi_b(t)$

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2D Analytic Signal

- Generalizing the analytic scale space from $1D$ to $2D$ can be framed as how to generalize the Poisson kernel to $2D$ space.
- For $2D$ signals, there are two choices:
  - On the upper-half space ($\mathbb{R}^3_+; \partial\mathbb{R}^3_+ = \mathbb{R}^2$):

$$P_s(x_1, x_2) := \frac{1}{2\pi} \frac{s}{\left(x_1^2 + x_2^2 + s^2\right)^{\frac{3}{2}}}; \quad s > 0$$

→ *Monogenic Scale Space*

  - On the bidisc ($\mathbb{D}^2 \subset \mathbb{C}^2; \partial_d \mathbb{D}^2 = \mathbb{T}^2$):

$$\begin{aligned} P_{r_1,r_2}(x_1, x_2) &:= P_{r_1}(x_1) P_{r_2}(x_2) \\ &= \prod_{k=1}^{2} \frac{1}{2\pi} \frac{1 - r_k^2}{1 - 2r_k \cos(x_k) + r_k^2}; \quad 0 < r_1, r_2 < 1 \end{aligned}$$

→ *2D Analytic Scale Space*

2D Analytic Signal

- For this talk, we will focus on the *2D analytic scale space* of the image $f : \mathbb{T}^2 \to \mathbb{R}$.
- Bülow (1999) proposed to use the imaginary units $\mathbb{i}$ and $\mathbb{j}$ of the quaternions $\mathbb{H}$, in order to distinguish the different conjugate harmonics, so we take our two complex variables to be $z_1 = r_1 e^{\mathbb{i} x_1}$ and $z_2 = r_2 e^{\mathbb{j} x_2}$, and define our *Cauchy kernel on $\mathbb{D}^2$* to take the form

$$C_{r_1,r_2}(x_1,x_2) := \left(P_{r_1}(x_1) + \mathbb{i} Q_{r_1}(x_1)\right)\left(P_{r_2}(x_2) + \mathbb{j} Q_{r_2}(x_2)\right).$$

  When $r_1 = r_2 = r$ we use the shorthand $C_r(x_1,x_2)$.
- The real component then represents the 2D Poisson kernel, and the $\mathbb{i}, \mathbb{j}$, and $\mathbb{k}(= \mathbb{i} \cdot \mathbb{j})$ components are its conjugate harmonics.
- For $2D$ signals, $U = C_r * f$ solves *the Riemann-Hilbert problem* on $\mathbb{D}^2$:

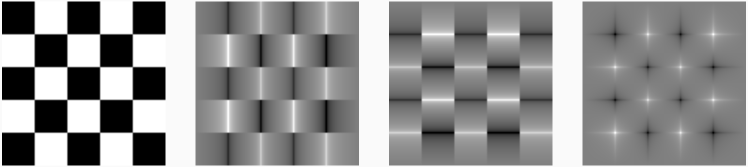
$$\begin{cases} \partial_{\bar{z}_1} U = 0 & \text{on } \mathbb{D}^2; \\ U \partial_{\bar{z}_2} = 0 & \text{on } \mathbb{D}^2; \\ \mathrm{Re}(U) = f & \text{on } \mathbb{T}^2. \end{cases}$$

2D Analytic Signal

- Analogously, we define the *2D analytic signal* on the $\mathbb{T}^2$ to be the limit of $C_{r_1,r_2}(\cdot,\cdot) * f$ as $r_1, r_2 \to 1$:

$$u(x_1,x_2) = f(x_1,x_2) + \mathbb{i}\mathscr{H}_1 f(\cdot,x_2)(x_1) + \mathbb{j}\mathscr{H}_2 f(x_1,\cdot)(x_2) + \mathbb{k}\mathscr{H}_T f(x_1,x_2).$$

- Here $\mathscr{H}_1, \mathscr{H}_2$ and $\mathscr{H}_T := \mathscr{H}_1\mathscr{H}_2$ denote the partial and total Hilbert transforms.



(a) f

(b) $\mathscr{H}_1 f$

(c) $\mathscr{H}_2 f$

(d) $\mathscr{H}_T f$

Polar Forms in $\mathbb{H}$

Given $q = q_0 + \mathbb{i} q_1 + \mathbb{j} q_2 + \mathbb{k} q_3 \in \mathbb{H}$,

- $\bar{q} = q_0 - \mathbb{i} q_1 - \mathbb{j} q_2 - \mathbb{k} q_3$, $|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$
- Polar form:

$$q = |q| \mathrm{e}^{\mu_q \phi_q}$$

  where

$$\mu_q = \frac{\mathbb{i} q_1 + \mathbb{j} q_2 + \mathbb{k} q_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}}, \; \phi_q = \arctan\left(\frac{\sqrt{q_1^2 + q_2^2 + q_3^2}}{q_0}\right).$$

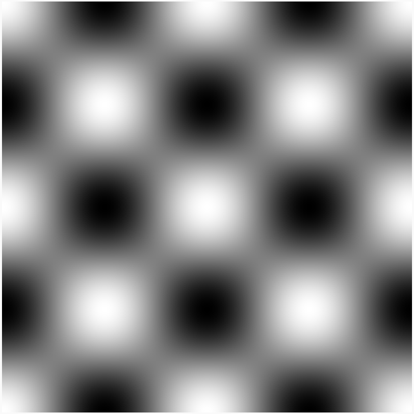
- *Euler angle form* as in Bülow (1999):

$$q = |q| \mathrm{e}^{\mathbb{i}\phi} \mathrm{e}^{\mathbb{k}\psi} \mathrm{e}^{\mathbb{j}\theta}$$

  for $\phi \in [-\pi, \pi), \theta \in [-\pi/2, \pi/2), \psi \in [-\pi/4, \pi/4]$.

2D Analytic Signal: An Example

- $f(x, y) = \cos(x - y) + \cos(x + y) = 2\cos(x)\cos(y)$, the "egg-tray" signal



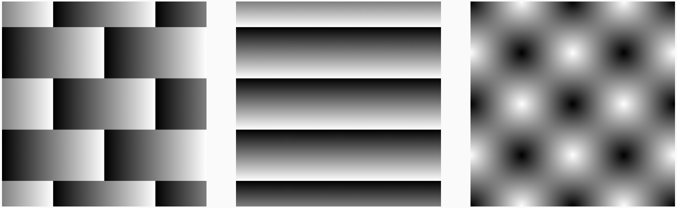
The egg-tray signal

2D Analytic Signal: An Example

As this signal is separable, we use what we know about the 1D Hilbert transform to compute:

$$\begin{aligned} u(x,y) &= 2\cos(x)\cos(y) + 2\mathbb{i}\sin(x)\cos(y) + 2\mathbb{j}\cos(x)\sin(y) + 2\mathbb{k}\sin(x)\sin(y) \\ &= 2e^{\mathbb{i}x}e^{\mathbb{j}y}, \end{aligned}$$

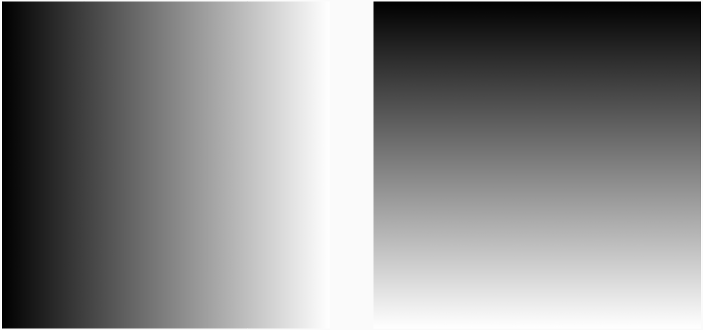
hence we have $|u(x,y)| = 2$, $\phi(x,y) = x$, $\theta(x,y) = y$, $\phi_q = \arctan\left(\frac{\sqrt{q_1^2+q_2^2+q_3^2}}{q_0}\right)$ where $q = u(x,y)$. We really recover $\phi(x,y) = x \mod \pi$, $\theta(x,y) = y \mod \pi/2$, due to phase wrapping:



(a) $\phi(x,y)$ (b) $\theta(x,y)$ (c) phase ϕ_q

2D Analytic Signal: An Example

We can then unwrap these two $1D$ phases easily using the standard 1D phase unwrapping algorithm, to achieve a nice monotonic $2D$ phase function:

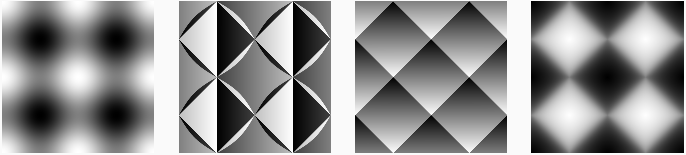


(a) $\phi(x, y) = x$ unwrapped

(b) $\theta(x, y) = y$ unwrapped

2D Analytic Signal: Comments

- If we instead consider a rotated egg-tray signal by $\pi/4$, we do not recover the true amplitude and phase of the signal as we do in the previous case. This is because the Hilbert transforms are directional, or *anisotropic*:



(a) $f(x,y)$ (b) $\phi(x,y)$ (c) $\theta(x,y)$ (d) ϕ_q

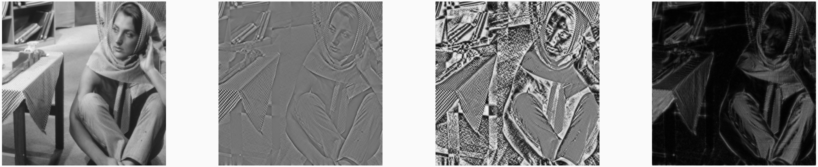
- This is a consequence of the *underlying geometry of* $\mathbb{D}^2$, if we are working with separable signals or products of orthogonal $1D$ signals, this tool can be quite useful.
- The 2D scale space allows us to control the scale of each of these signals independently with good interpretation, which we cannot do in the related monogenic or Gaussian scale space, for instance.

2D Analytic Scale Space of a Real Image: An Example

We may use this 2D scale space to create a bandpass filter by taking the difference of two Cauchy kernels:

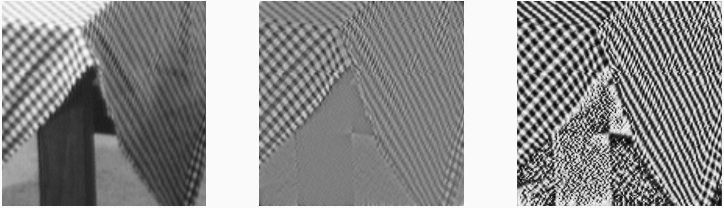
$$DoC_{r_1,r_2} := C_{r_1}(x_1, x_2) - C_{r_2}(x_1, x_2)$$

for $0 \le r_2 < r_1 \le 1$ and compute the IAP representation of real images:



(a) Barbara (b) $\mathrm{Re}(DoC_{1,0.95})$ (c) phase ϕ_q (d) $|DoC_{1,0.95}|$

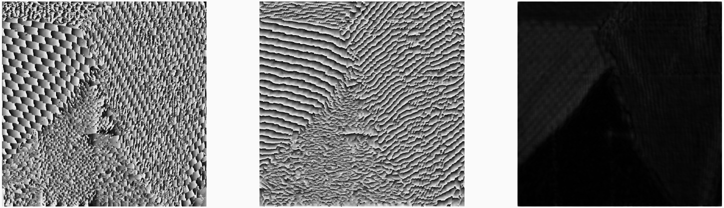
2D Analytic Scale Space of a Real Image: An Example



(a) table corner

(b) $\mathrm{Re}\left(DoC_{1,0.95}\right)$

(c) phase ϕ_q



(a) $\phi(x_1, x_2)$

(b) $\theta(x_1, x_2)$

(c) $|DoC_{1,0.95}|$

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- In $1D$, the *Blaschke factorization* yields a satisfactory IAP representation of a signal, and the resulting phase function can be used for various image processing tasks, e.g., phase unwrapping.
- In $2D$, the situation is not so clear: the $2D$ analytic signal can be formed in various ways; we showed that at least with a synthetic image, we can produce a *quaternion-valued signal* that captures instantaneous amplitude and phase accurately.
- The *2D analytic scale space* provides a multiscale instantaneous amplitude and phase decomposition which could be utilized to form smooth phase functions for sufficiently band-limited signals.
- There are many more things to do in $2D$!!

Thank you!