Minimal Convex Combinations of Three Sequential Laplace-Dirichlet Eigenvalues

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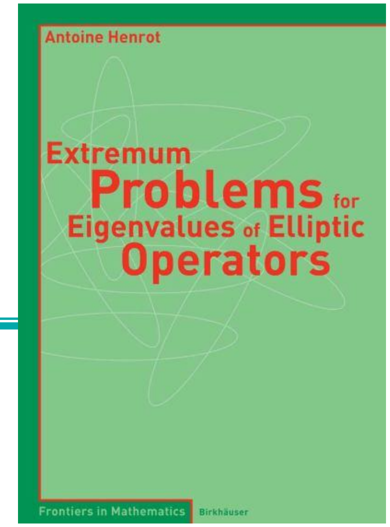
Outline

1. Introduction to shape optimization on eigenvalue problems
2. Review of Previous Theoretical and Numerical Results
3. Our Theoretical and Numerical Results
4. Conclusion and Future Work

Shape Optimization on Eigenvalue Problems



- Goal: Minimize a certain design objective

$$\inf_{\Omega}\{\, f(\Lambda(\Omega)) : \Omega \textit{ is open set in } R^d, |\Omega| = 1 \,\}$$

where $\Lambda(\Omega) = \{\lambda_k(\Omega)\}_{k=1}^{\infty}$ are the Laplace-Dirichlet eigenalue satisfying

$$\begin{cases} -\Delta\psi_j(x) = \lambda_j\psi_j & x \in \Omega, \\ \psi_j(x) = 0 & x \in \partial\Omega, \end{cases}$$

f is a real-valued function on the eigenvalue sequences, and $|\cdot|$ denotes the Lebesgue measure. [Henrot 2006]

Three Sequential Eigenvalue Problem

We consider the following (α, β) parameterized optimization problem:

$$C_{\alpha,\beta}^{j*} = \inf_{\Omega \in \mathcal{A}} C_{\alpha,\beta}^{j}(\Omega) \quad \text{and} \quad \hat{\Omega}_{\alpha,\beta}^{j} = \{\Omega \in \mathcal{A} \colon C_{\alpha,\beta}^{j}(\Omega) = C_{\alpha,\beta}^{j*}\},$$

where

$$T := \{(\alpha, \beta) \in \mathbb{R}^2 \colon \alpha \geq 0, \beta \geq 0, \alpha + \beta \leq 1\},$$

$$\mathcal{A} := \{\Omega \subset \mathbb{R}^2 \colon \Omega \text{ quasi-open and } |\Omega| \leq 1\},$$

$$C_{\alpha,\beta}^{j}(\Omega) := \alpha\lambda_j(\Omega) + \beta\lambda_{j+1}(\Omega) + (1 - \alpha - \beta)\lambda_{j+2}(\Omega),$$

$$(\alpha, \beta) \in T \text{ and } \Omega \in \mathcal{A}.$$

Reduced Eigenvalue Problem

$$C^j_{\alpha,\beta}(\Omega) := \alpha\lambda_j(\Omega) + \beta\lambda_{j+1}(\Omega) + (1-\alpha-\beta)\lambda_{j+2}(\Omega)$$

1. Minimization of a single eigenvaue

$$\min_{\Omega} \lambda_j = \min_{\Omega\in\mathcal{A}} C^j_{1,0}(\Omega)$$

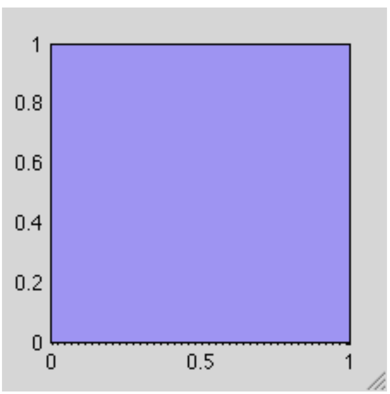
2. A convex combination of two sequential Laplace-Dirichlet eigenvalues

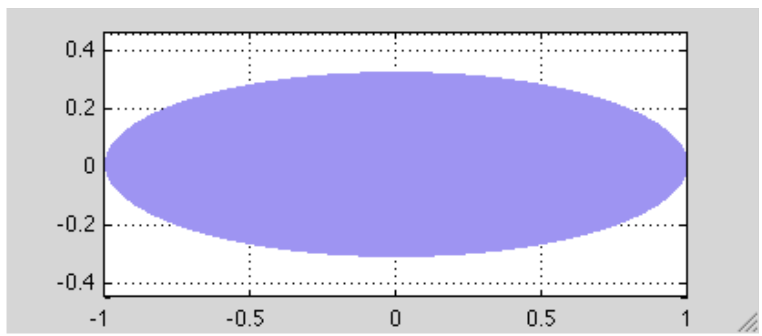
$$\min_{\Omega\in\mathcal{A}} C^j_{\alpha,1-\alpha}(\Omega) \quad \text{for } \alpha \in [0,1],$$

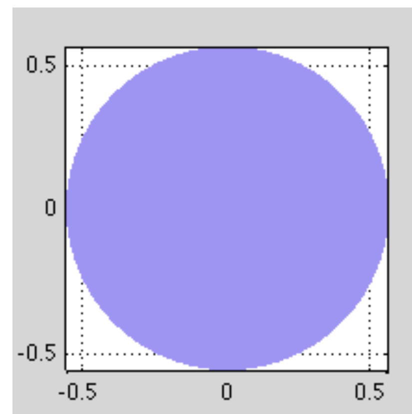
Minimization of the first eigenvalue

$$\Omega \to \lambda_1(\Omega),\ |\Omega| = 1$$

$$\min_{\Omega} \lambda_1$$







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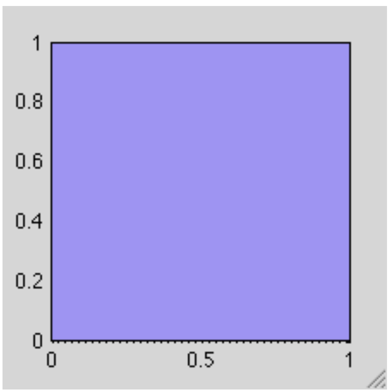
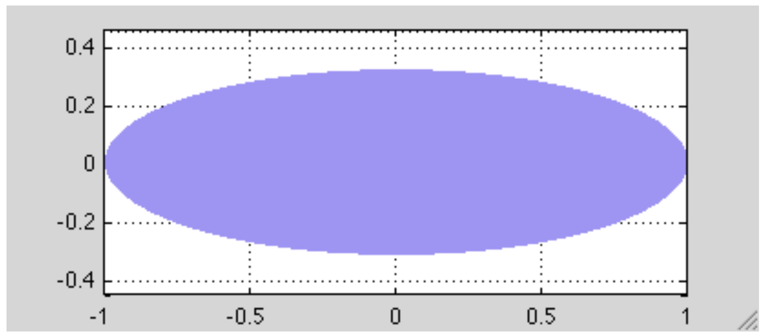
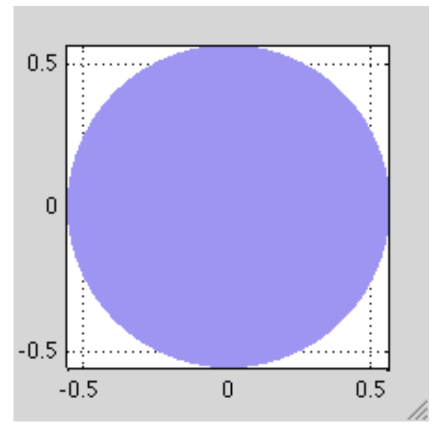
Minimization of the first eigenvalue



$$\lambda_1 = 2\pi^2 \approx 19.74$$
$$\lambda_2 = 5\pi^2 \approx 49.35$$
$$\lambda_3 = 5\pi^2 \approx 49.35$$
$$\lambda_4 = 8\pi^2 \approx 78.96$$



$$\lambda_1 \approx 30.33$$
$$\lambda_2 \approx 44.10$$
$$\lambda_3 \approx 61.69$$
$$\lambda_4 \approx 83.36$$



$$\lambda_1 \approx 18.16$$
$$\lambda_2 \approx 46.12$$
$$\lambda_3 \approx 46.12$$
$$\lambda_4 \approx 82.86$$

Theoretical Results

"If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle." **—Lord Rayleigh (1877)**

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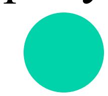
- 1923: Faber and Krahn prove that a ball uniquely minimizes first eigenvalue among domains of unit measure. $\lambda_1 \geq \pi j_{0,1}^2$ 

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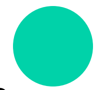
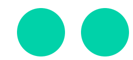
- 1923: Faber and Krahn prove that a ball uniquely minimizes first eigenvalue among domains of unit measure. $\lambda_1 \geq \pi j_{0,1}^2$ 
- 1926: Krahn proves that two disjoint balls of equal measure uniquely minimizes the second eigenvalue among domains of unit measure. $\lambda_2 \geq 2\pi j_{0,1}^2$ 

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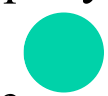
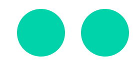
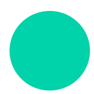
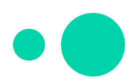
- 1923: Faber and Krahn prove that a ball uniquely minimizes first eigenvalue among domains of unit measure. $\lambda_1 \geq \pi j_{0,1}^2$ 
- 1926: Krahn proves that two disjoint balls of equal measure uniquely minimizes the second eigenvalue among domains of unit measure. $\lambda_2 \geq 2\pi j_{0,1}^2$ 
- 1994 Wolf and Keller show that any minimizer for third eigenvalue is connected and a ball is a local minimizer among star-shaped, bounded domains with smooth boundary. 
- 1994 The minimizer of fourth eigenvalue is disjoint union of two balls with radii which have the ratio $j_{0,1} / j_{1,1}$ 
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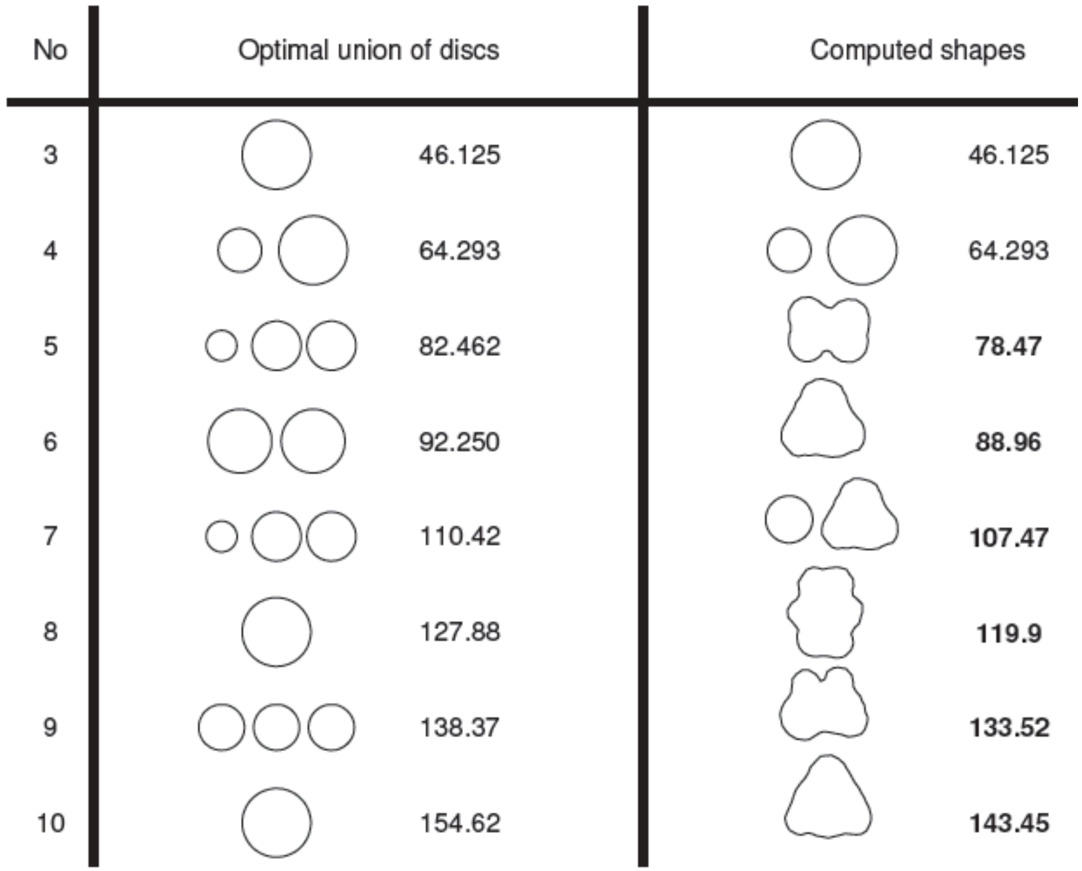
- 2000 Bucur and Henrot prove that the minimizer of third eigenvalue exists
- 2012 Bucur, Mazzoleni and Pratelli prove the infimum for minimization of kth eigenvalues exists and every minimizer has finite perimeter

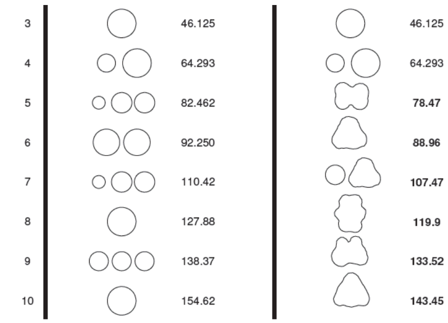
Numerical Result I

Oudet (2004)

1. Level set method
2. The k-th eigenvalue of the minimizer is multiple when k is greater than one



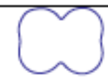
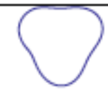
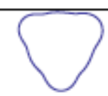
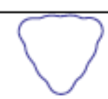
No	Optimal union of discs	Computed shapes
3	46.125	46.125
4	64.293	64.293
5	82.462	**78.47**
6	92.250	**88.96**
7	110.42	**107.47**
8	127.88	**119.9**
9	138.37	**133.52**
10	154.62	**143.45**



Numerical Result II

Antunes & Freitas (2012)

1. Eigenvalues computed via meshless method
2. Domains parameterized using Fourier coefficients
3. k=7 result is improved
4. k=13 minimizer is not symmetric

i	Ω	multiplicity	λ_i^*	Oudet's result
5		2	**78.20**	78.47
6		3	**88.52**	88.96
7		3	**106.14**	107.47
8		3	**118.90**	119.9
9		3	**132.68**	133.52
10		4	**142.72**	143.45
11		4	**159.39**	-
12		4	**172.85**	-
13		4	**186.97**	-
14		4	**198.96**	-
15		5	**209.63**	-

Minimization of Two Sequential Eigenvalues

Ashbaugh and Benguria (1991),

Wolf and Keller (1994),

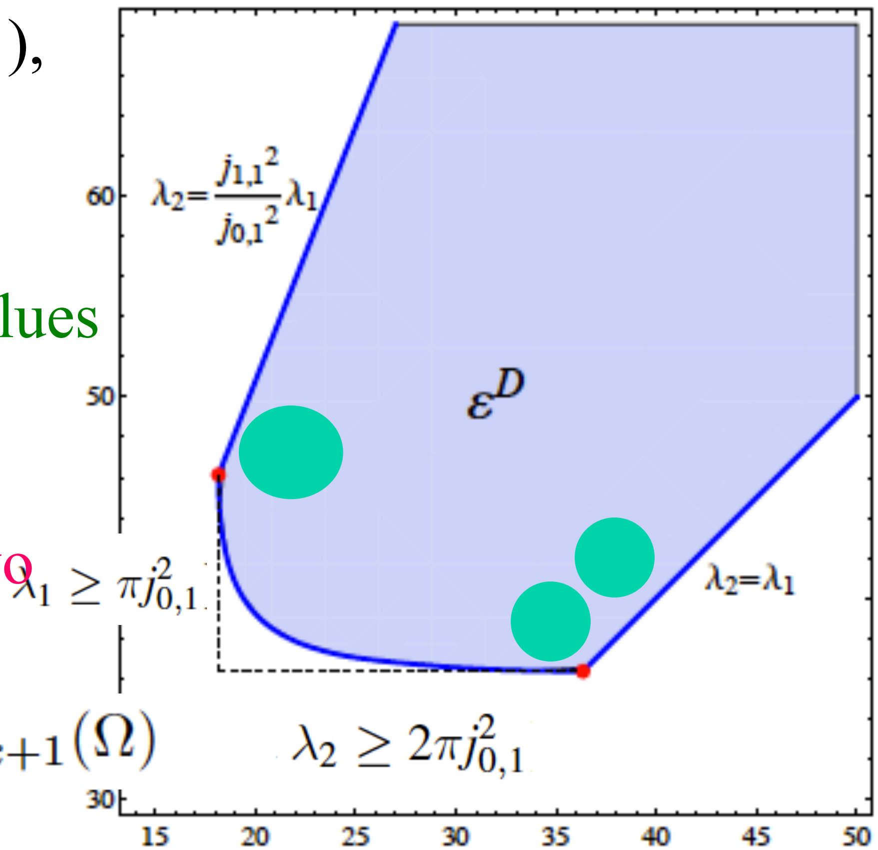
Antunes and Henrot (2010).

Range of the first two eigenvalues

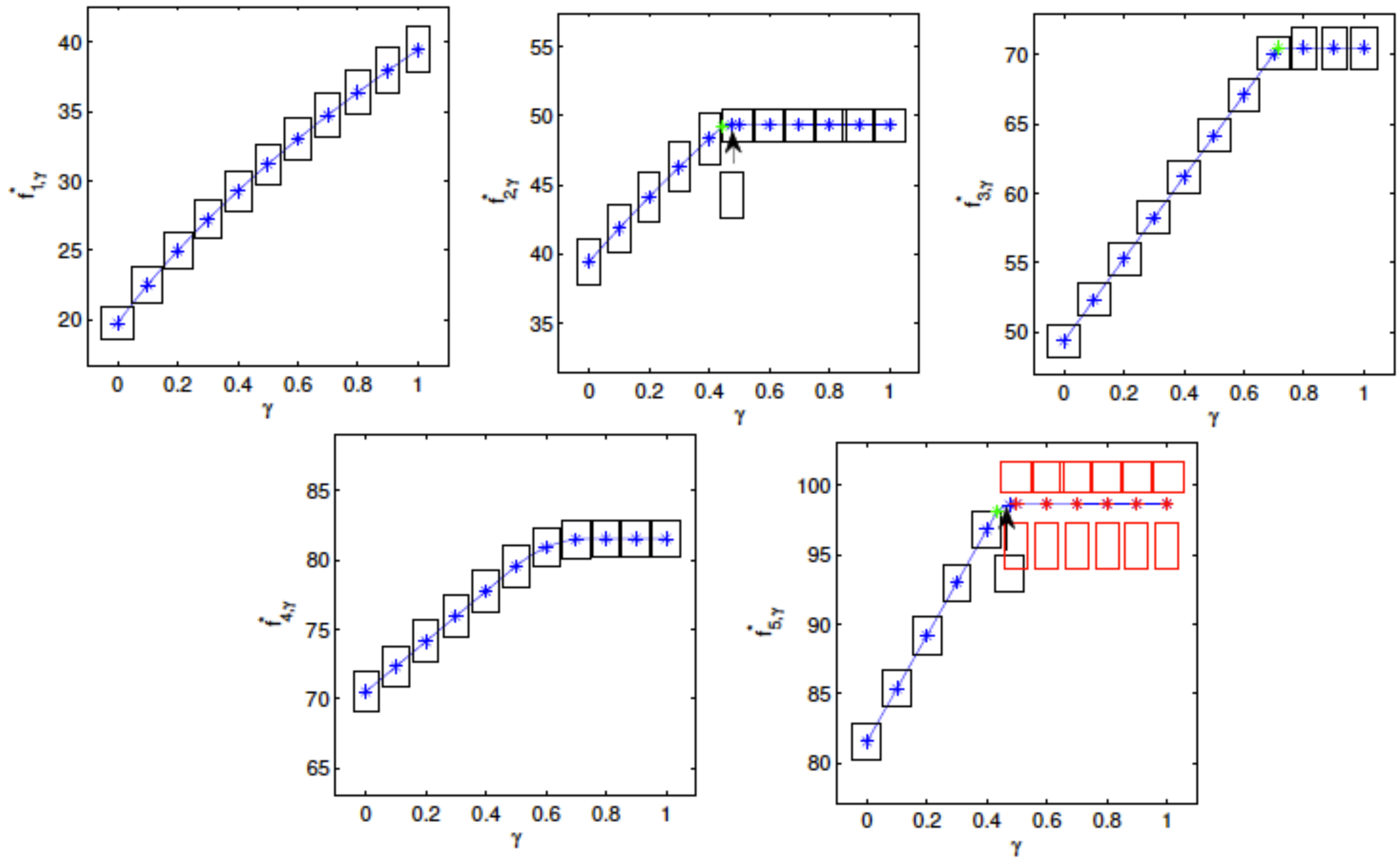
1. Curve determined by minimizers of convex combination of the first two eigenvalues

$$C_{k,\gamma}(\Omega) := (1-\gamma)\,\lambda_k(\Omega) + \gamma\,\lambda_{k+1}(\Omega)$$

2. Topological change

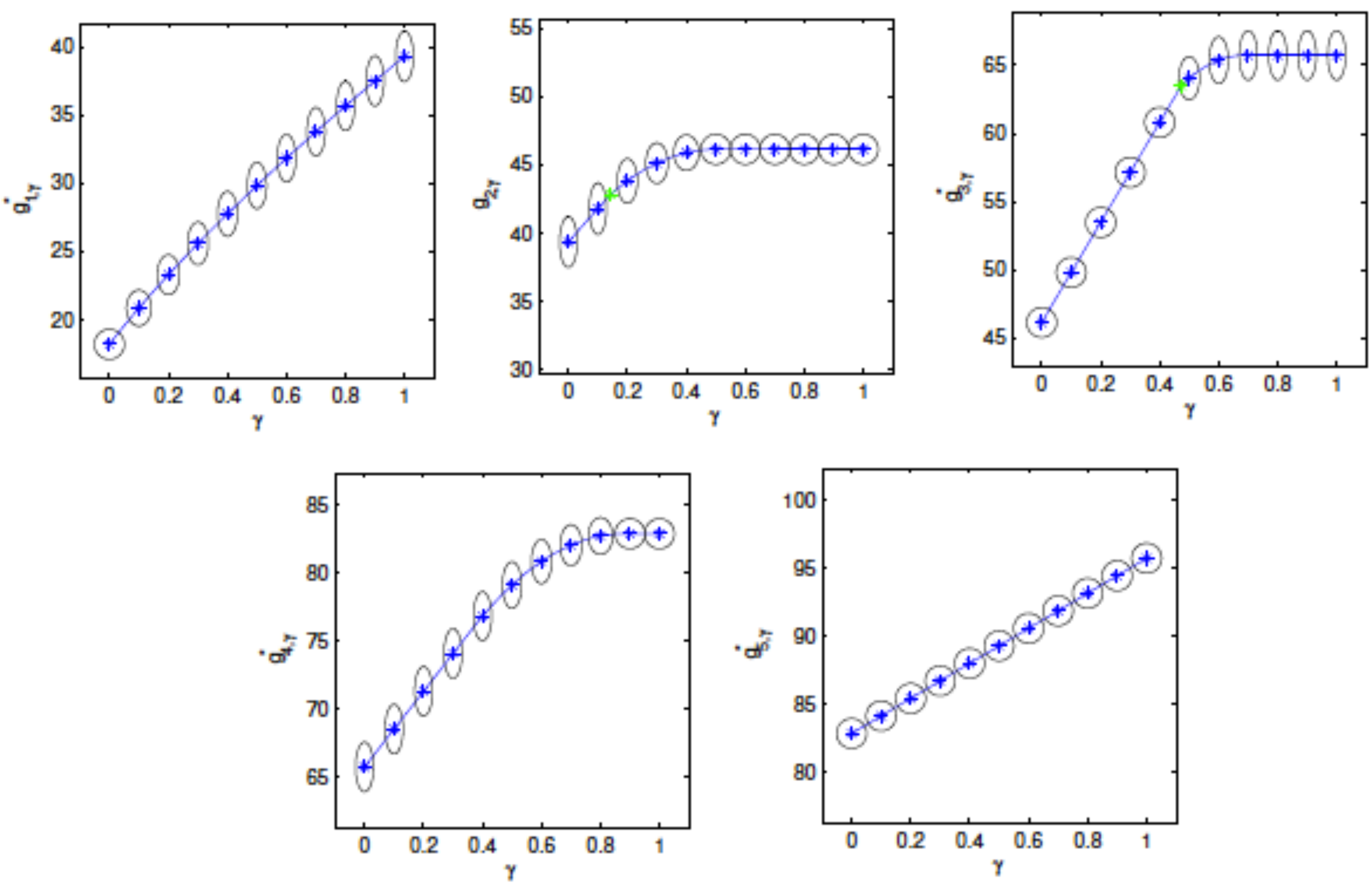


Minimization of Two Sequential Eigenvalues among rectangular shapes

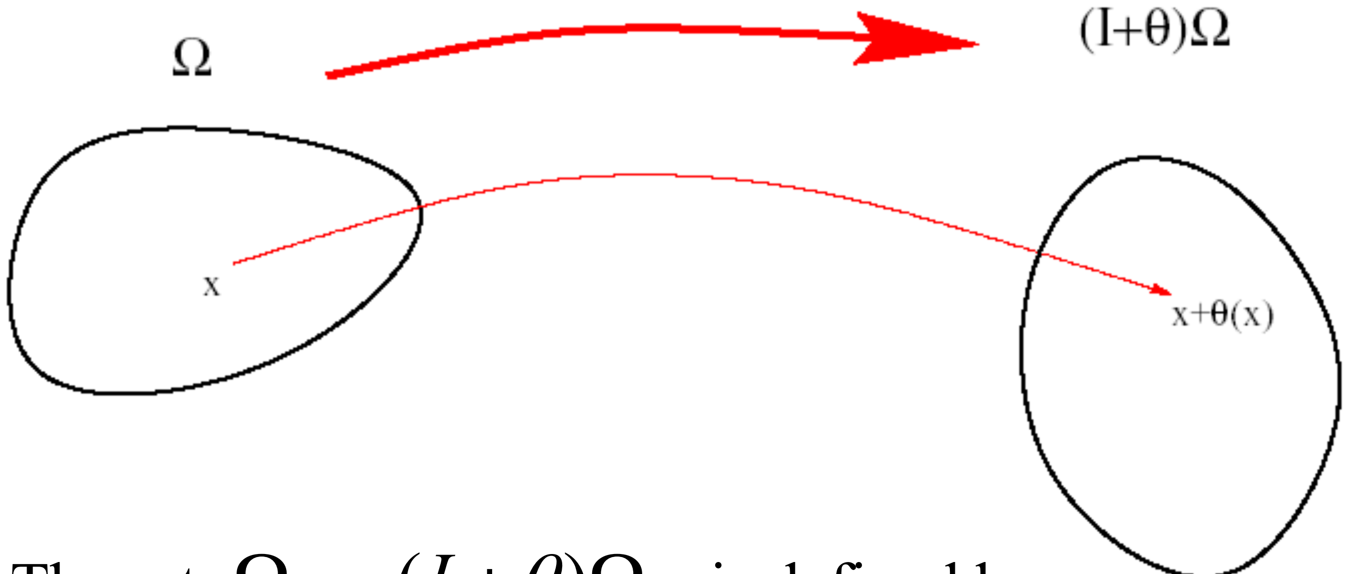


$$C_{k,\gamma}(\Omega) := (1-\gamma)\,\lambda_k(\Omega) + \gamma\,\lambda_{k+1}(\Omega)$$

Minimization of Two Sequential Eigenvalues among elliptical shapes



Shape Mapping



- The set $\Omega_\theta = (I+\theta)\Omega$ is defined by

$$\Omega_\theta = \{x + \theta(x) \mid x \in \Omega\}$$

- The vector field $\theta(x)$ is the displacement of Ω .

Shape Derivative

- Framework of Murat-Simon:
- Let Ω be a reference domain. Consider its variations
- $\Omega_\theta = (I+\theta)\Omega$ with $\theta \in W^{1,\infty}(R^N; R^N)$

- **Definition**: the shape derivative of $F(\Omega)$ at Ω is the Frechet differential of $\theta \rightarrow F((I+\theta)\Omega)$ at 0 .

$$d_s F(\Omega)(\theta) = \lim_{h\to 0} \frac{F((I+h\theta)\Omega) - F(\Omega)}{h}$$

Numerical Approach for general shapes

$$C_{k,\gamma}(\Omega) := (1-\gamma)\,|\Omega|\,\lambda_k(\Omega) + \gamma\,|\Omega|\,\lambda_{k+1}(\Omega)$$

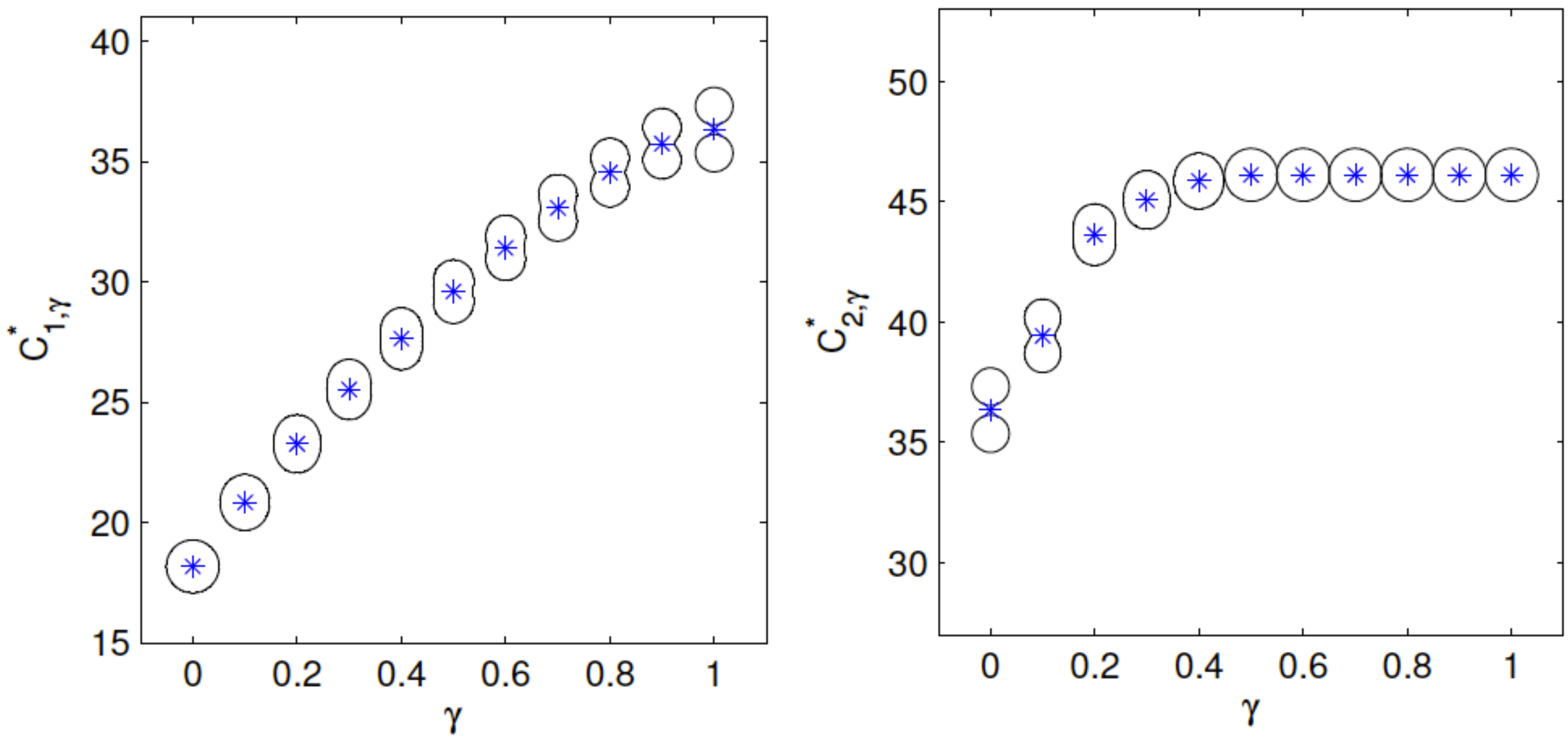
The Level Set Approach

$$\phi_t + V_n(\mathbf{x})|\nabla\phi| = 0$$

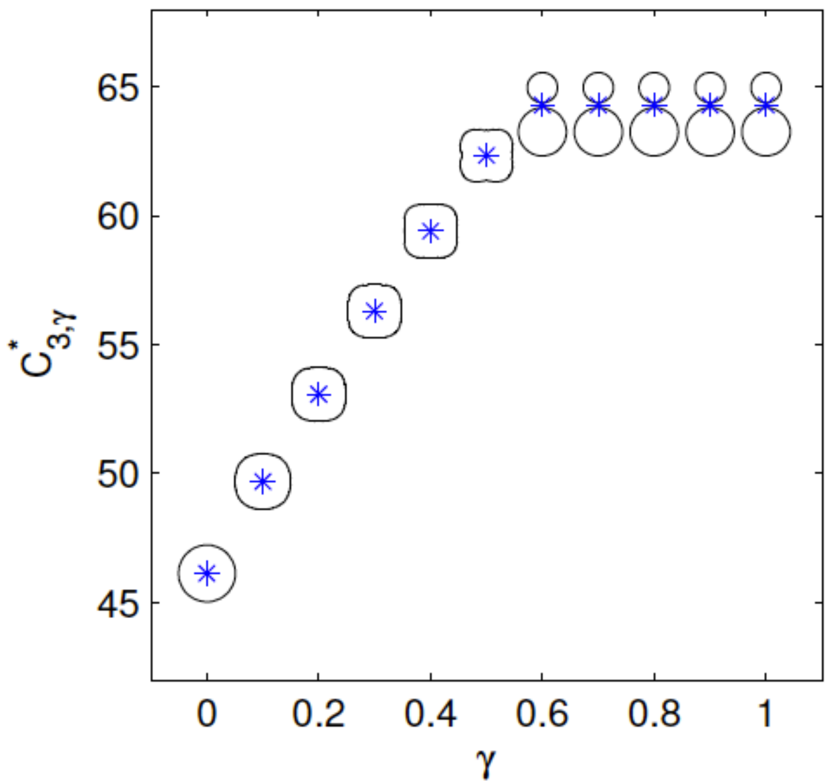
Velocity: (shape derivative)

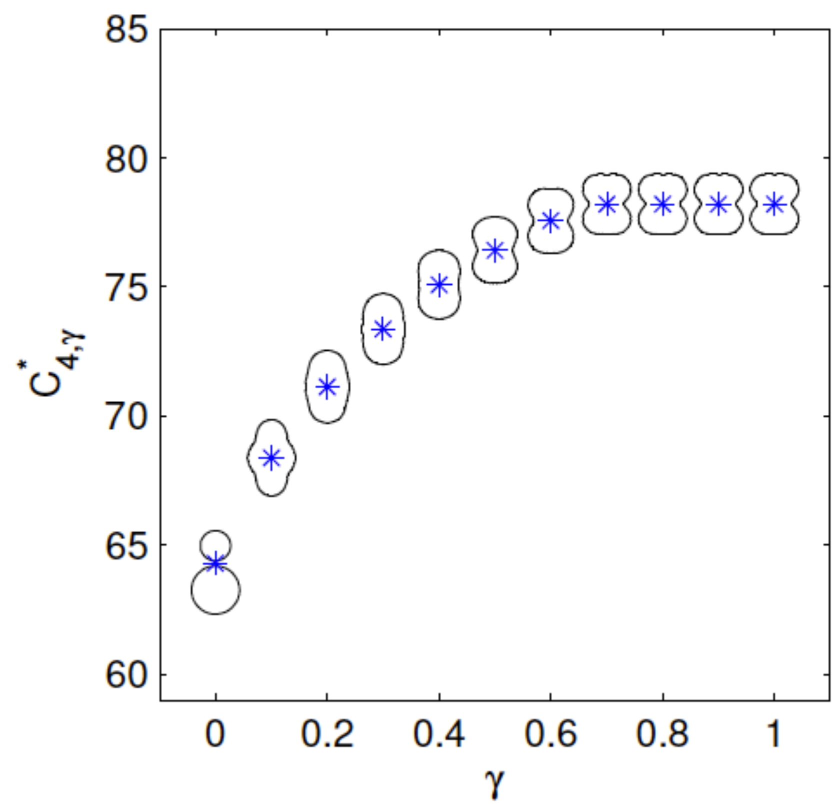
$$|\Omega|\left((1-\gamma)\,|\partial_n\psi_k|^2 + \gamma\,|\partial_n\psi_{k+1}|^2\right) - ((1-\gamma)\lambda_k + \gamma\lambda_{k+1})$$

Minimization of Two Sequential Eigenvalues among general shapes

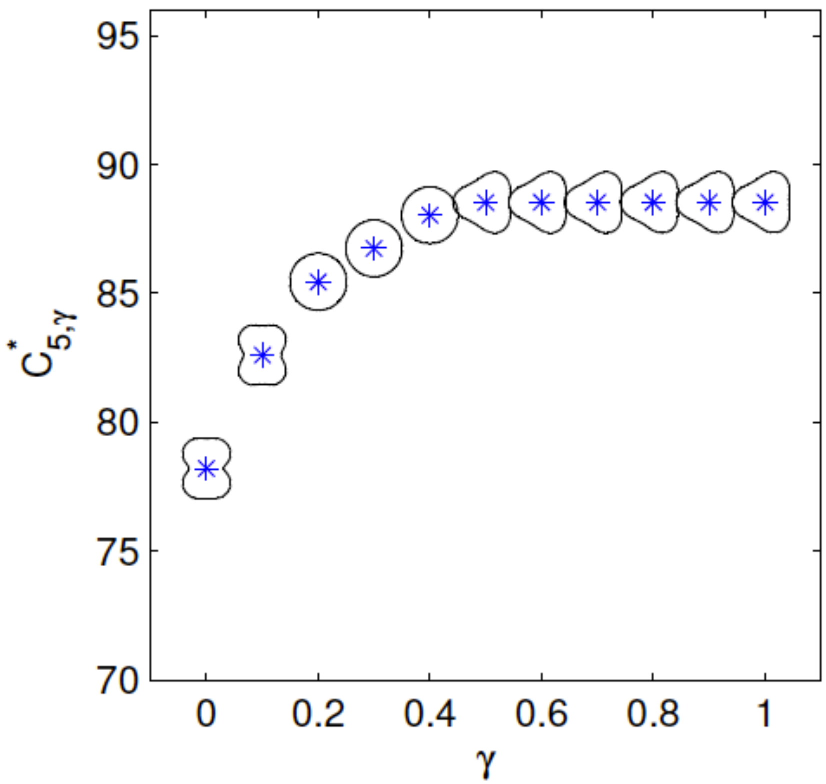


Minimization of Two Sequential Eigenvalues among elliptical shapes





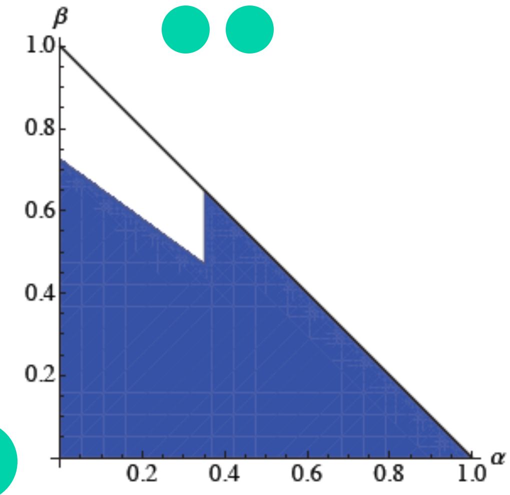
Minimization of Two Sequential Eigenvalues among elliptical shapes



1. The number of connected components of the minimizer varies with the convex combination parameter
2. For k=2:5, the optimizer for $\gamma=1$ is also the optimizer on an interval $\gamma \in [1-\delta, 1]$ for some δ
3. $C^{j,*}_{1-\gamma,\gamma}$ is non-decreasing Lipschitz continuous, and concave.

Minimization of
Three Sequential Eigenvalues

Theorem: Iversen and Pratelli (2012),



(a) Any minimizer is connected for each of the following cases:

(i) $\alpha + \beta = 1, \alpha > 0,$

(ii) $0.35 \lesssim \alpha \leq 1$

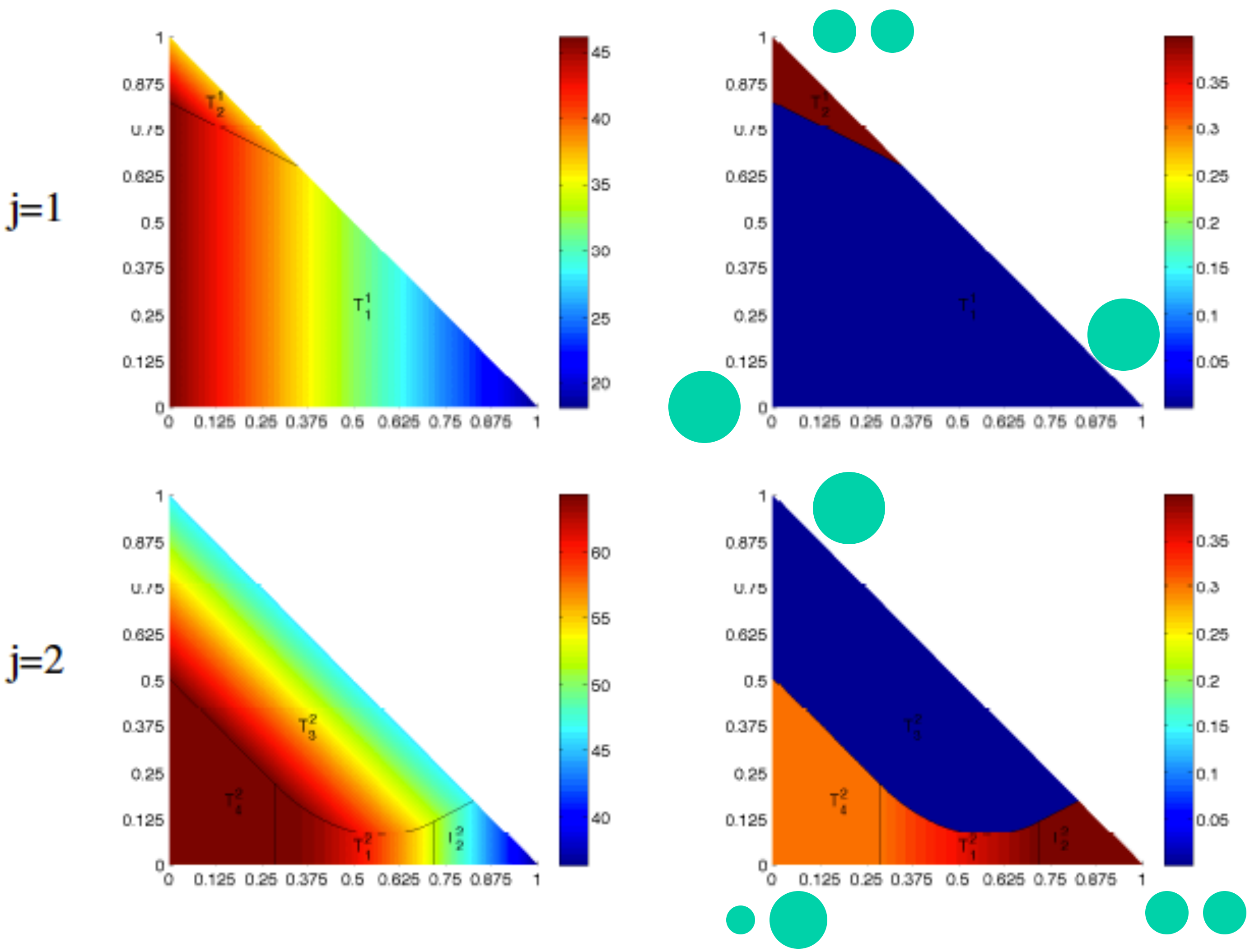
(iii) $0 \leq \beta \lesssim 0.725(1 - \alpha)$

(b) Any disconnected minimizer, Ω, satisfies $\lambda_1(\Omega) = \lambda_2(\Omega)$ and has exactly two components.

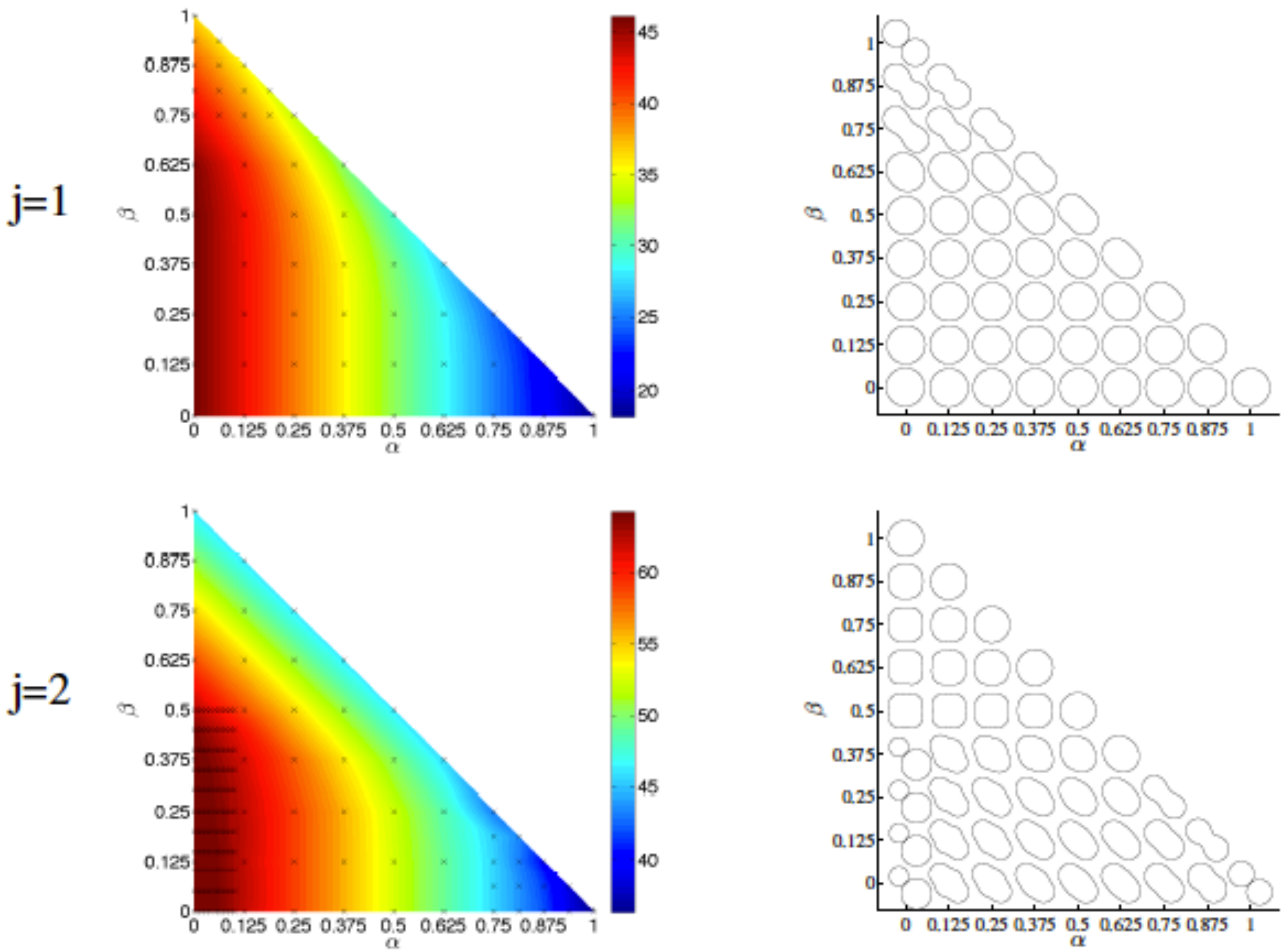
(c) If any minimizer is connected for $\alpha = 0$ and each $\beta \in [0, 1)$, then any minimizer is connected unless $\beta = 1$.

Iversen + Pratelli conjecture that the minimizer is connected unless $\beta = 1$.

Minimization of Three Sequential Eigenvalues among union of balls



Minimization of Three Sequential Eigenvalues among general shapes



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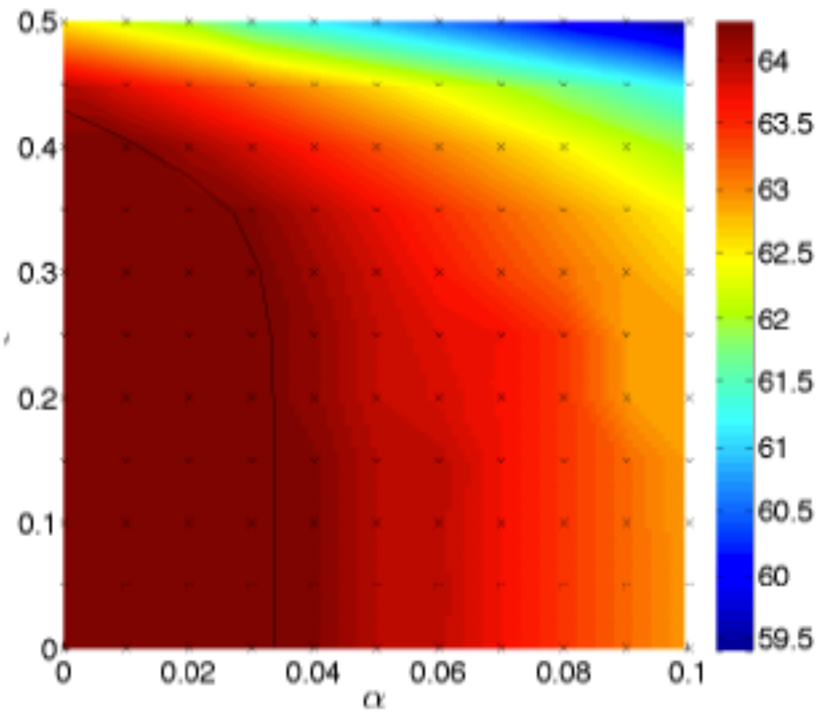

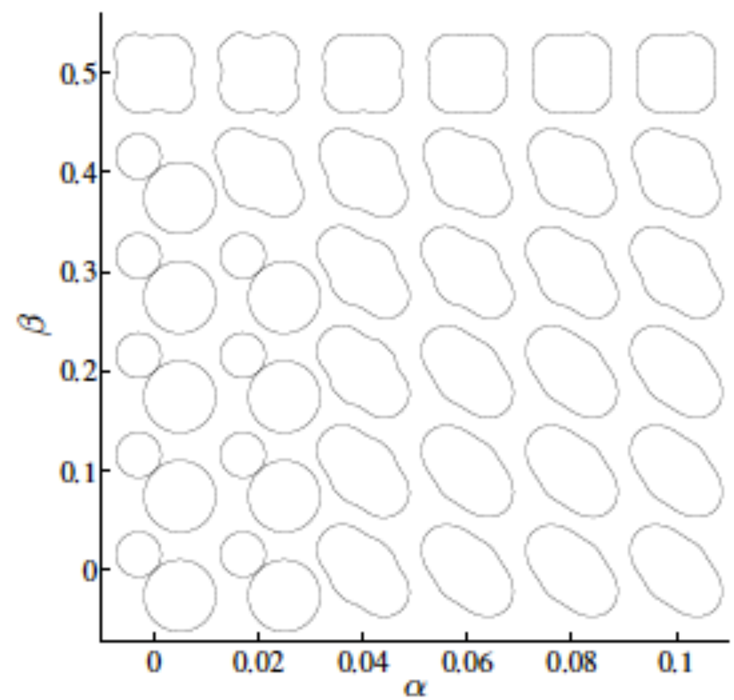

Minimization of Three Sequential Eigenvalues among general shapes

Theorem. For for the (α, β)-set

$$\{(\alpha, \beta) \in T \colon \alpha + 2\beta \leq 1\},$$

the ball is a local minimizer of $C^1_{\alpha,\beta}$ over the admissible class of equal measure, star-shaped, bounded domains with smooth boundary.

Idea of the proof: (1) perturbation of the domain

$$R(\theta, \epsilon) := 1 + \epsilon \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} + \epsilon^2 \sum_{k=-\infty}^{\infty} b_k e^{ik\theta} + O(\epsilon^3), \quad a_n = \overline{a_{-n}} \text{ and } b_n = \overline{b_{-n}}.$$

(2) asymptotic analysis

If $a_2 \neq 0$,

$$C^1_{\alpha,\beta}(\Omega_\epsilon) = \pi \left[\alpha j_{0,1}^2 + (1-\alpha) j_{1,1}^2\right] + 2\epsilon\pi j_{1,1}^2 (1 - \alpha - 2\beta) |a_2| + O(\epsilon^2)$$

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Idea of the proof: (continue)

and if $a_2 = 0$,

$$C^1_{\alpha,\beta}(\Omega_\epsilon) = \pi\left[\alpha j^2_{0,1} + (1-\alpha) j^2_{1,1}\right] + A\alpha\epsilon^2 + B(1-\alpha)\epsilon^2 + (1-\alpha-2\beta)C\epsilon^2 + O(\epsilon^3)$$

where

$$A = 4\pi j^2_{0,1} \sum_{n=1}^{\infty} \left(1 + j_{0,1}\frac{J'_n(j_{0,1})}{J_n(j_{0,1})}\right)|a_n|^2$$

$$B = 2\pi j^2_{1,1} \sum_{\ell} \left(1 + j_{1,1}\frac{J'_{\ell-1}(j_{1,1})}{J_{\ell-1}(j_{1,1})}\right)|a_\ell|^2$$

$$C = 2\pi j^2_{1,1}\left| b_2 - \sum_{\ell}\left(\frac{1}{2} + j_{1,1}\frac{J'_\ell(j_{1,1})}{J'_\ell(j_{1,1})}\right) a_{1+\ell}a_{1-\ell}\right|.$$

Future Directions

- Study Laplace eigenvalue problem with general boundary conditions
- Study BiLaplace eigenvalue problem with general boundary conditions
- Study Laplace-Beltrami eigenvalue problems for general manifold
- Study BiLaplace-Beltrami eigenvale problems for general manifold

The End

Thank you for your attention!!

Questions??