Applications of Laplace-Beltrami Spectrum via Conformal Deformation

Rongjie Lai

Department of Mathematics, University of California, Irvine

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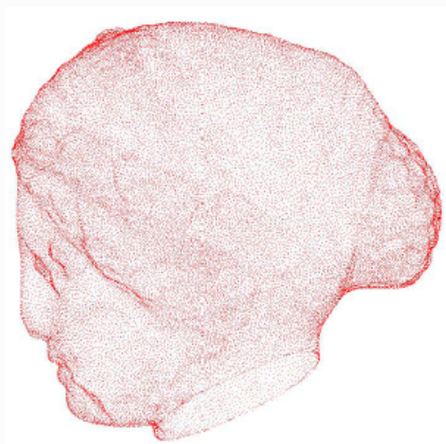
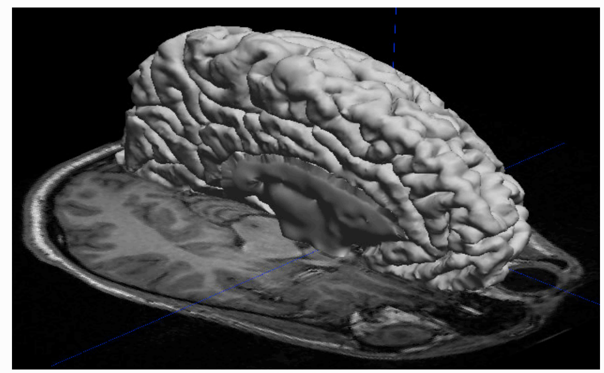
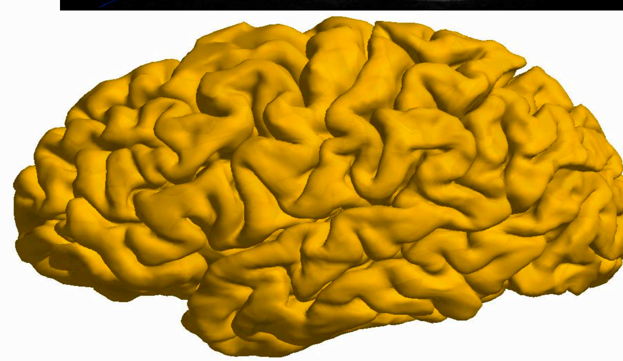
Motivation and Problems

Portable 3D scanner

Magnetic Resonance scanner







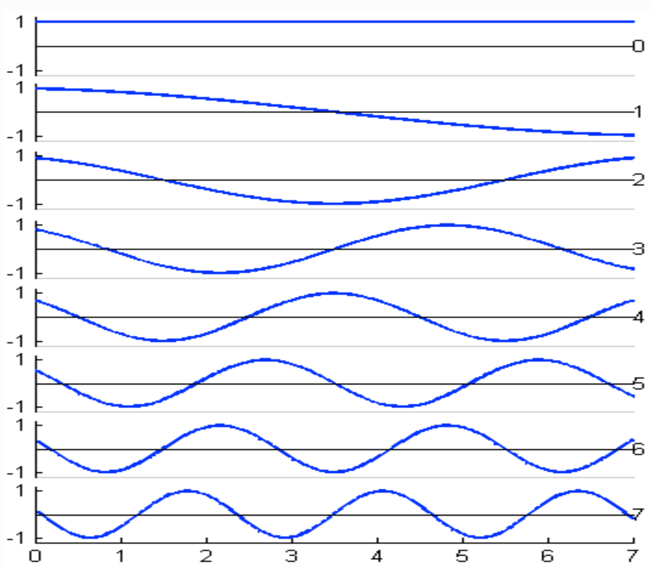
Laplace-Beltrami Eigen-system

A Laplace-Beltrami (LB) eigen-system of $(\mathcal{M}, g)$:

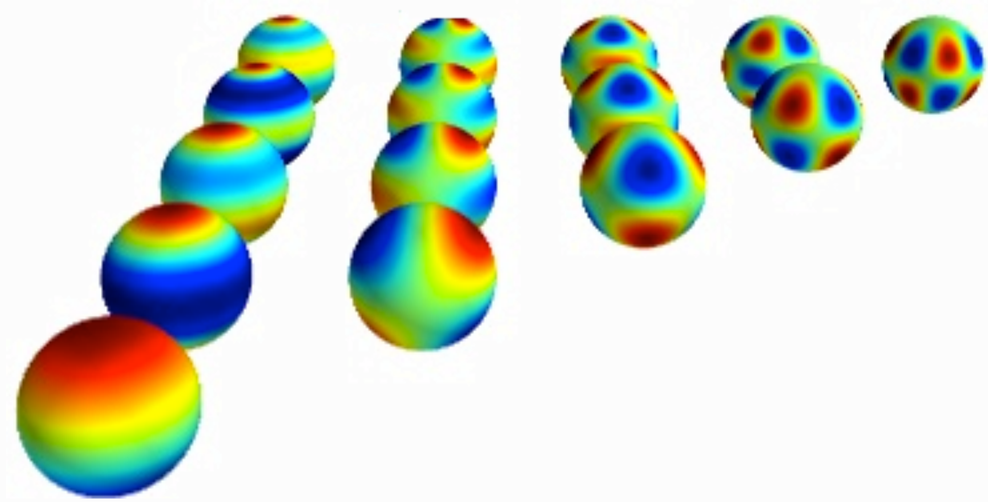
$$\begin{cases} \Delta_{\mathcal{M}}\phi_k = -\lambda_k \phi_k, & k = 1, 2, \cdots, \\ \partial_\nu \phi_k(x) = 0, & x \in \partial\mathcal{M} \end{cases}$$

- $\Delta_{\mathcal{M}}\phi_k = \frac{1}{\sqrt{G}}\partial_{x^i}(\sqrt{G}g^{ij}\partial_{x^j}\phi_k)$ is the LB operator of $(\mathcal{M}, g)$
- $\sigma(\mathcal{M}, g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ are eigenvalues of $\Delta_{\mathcal{M}}$ on $(\mathcal{M}, g)$.
- The corresponding $\phi_0, \phi_1, \phi_2, \cdots$ are called eigenfunctions.

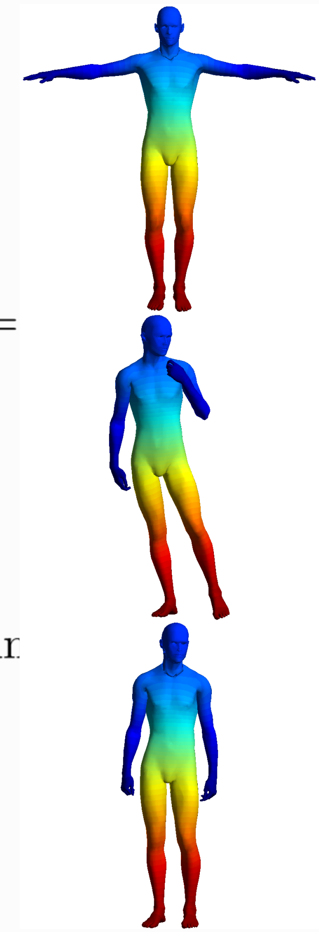
Example: Fourier basis and spherical harmonics



Fourier basis


Spherical harmonics

Computation of Laplace-Beltrami Eigen-system

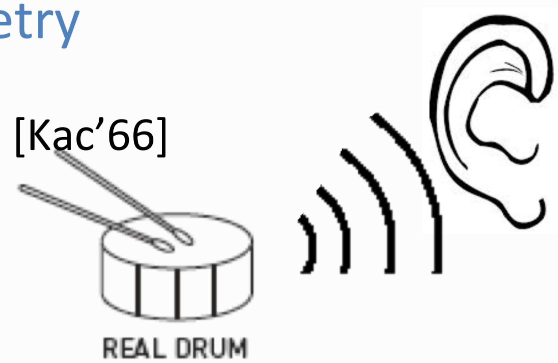


N=50

- Implicit approach: Level set method $\qquad \mathcal{M} = f^{-1}(0) \subset R^3$
  - J. Brandman'08, The first few LB eigenfucntions $\Longleftrightarrow$ a PDE(the tangential component) + an ODE (normal component).
  - Gao-Lai-Shi'10 propose to approximate the LB eigenproblem on the narrow band $\mathcal{M}_\delta = f^{-1}([-\delta, \delta])$. If $\dfrac{\pi^2}{4\delta^2} > \lambda_k(\mathcal{M})$, then $\lambda_i(\mathcal{M}_\delta) = \lambda_i(\mathcal{M}), \quad i = 0, \cdots, k$
- Closest point method. [C. Macdonald-J. Brandman-S. Ruuth]
- Triangle mesh based method
  - Finite difference method to approximate the LB operator. [Taubin'00,Desbrun Meyer-Schroder-Bar'02, Xu'04], : $\Delta_{\mathcal{M}} f(v_i) \approx \sum_{j \in N(i)} \omega_{ij} \left(f(v_j) - f(v_i)\right)$
  - Finite element method based on the weak formula [M. Reuter'06, A.Qiu'06]
- Point cloud based method, PDEs on high dimensional manifolds.
  - Diffusion based method, only for Laplacian [Belkin-Sun-Wang,Coifman-Lafon]
  - Two systematic methods, can be applied for other PDEs. [Lai-Liang-Zhao]

Relation between LB Eigen-system and surface geometry

- Inverse spectral problem: Can one hear the shape of a drum? [Kac'66]



  - Heat trace asymptotic expansion:

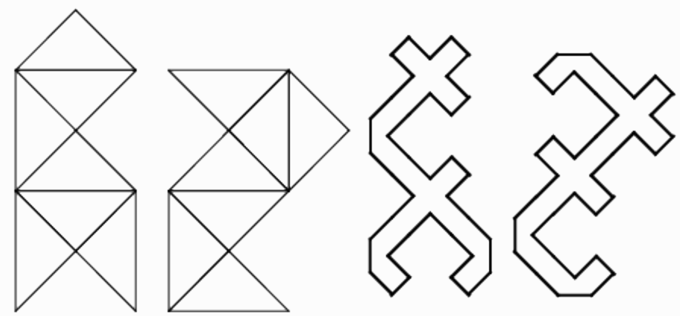
$$Z(t) = \int_{\mathcal{M}} K(t,x,x)\mathrm{d}v(x) = \frac{1}{4\pi t}\sum_i c_i t^{i/2}$$

$$c_0 = area(\mathcal{M}), \quad c_1 = -\frac{\sqrt{\pi}}{2} length(\partial\mathcal{M}), \quad c_2 = \frac{1}{3}\int_{\mathcal{M}} K - 1/6\int_{\partial\mathcal{M}} J$$

where $K(t,x,y) = \sum_i e^{-\lambda_i t}\phi_i(x)\phi_i(y)$, K is the Gauss curvature of $\mathcal{M}$ and J is the mean curvature of $\partial\mathcal{M}$ in $\mathcal{M}$. Moreover, if $\mathcal{M}$ is a closed surface with Euler number $\chi(\mathcal{M})$, then $c_2 = 2\chi(\mathcal{M})/3$.[H. Mckean, I. Singer'67]

  - Isospectral surfaces. LB eigenvalues can not uniquely determine a surface.

[Milnor,Sunada, Gordon-Webb-Wolpert et al.]



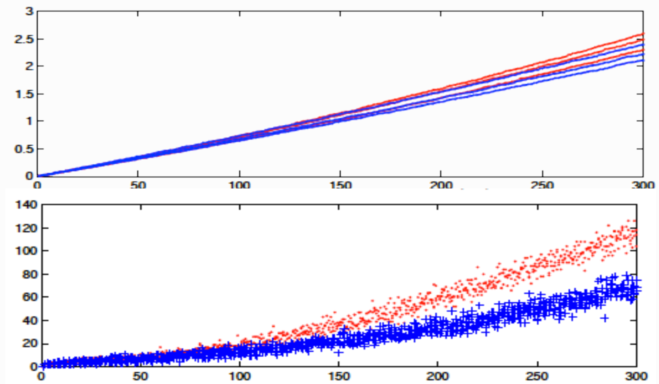
- Asymptotical behavior [Weyl, 1910]

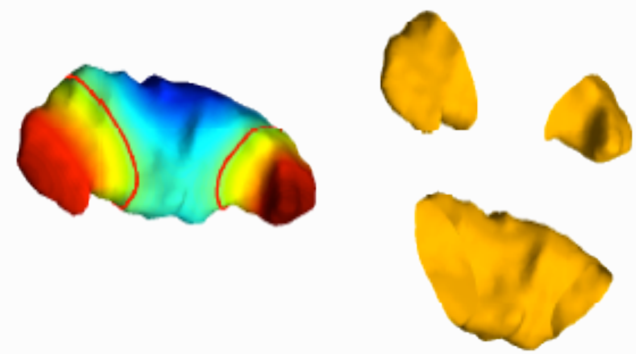
$$\lambda_k \sim \frac{4\pi^2 k^{2/d}}{(C_d Vol(\mathcal{M}))^{2/d}} \text{ as } k \to \infty, \text{ here } C_d = vol(d\text{-ball})$$

- Shape DNA[Reuter'06], Heat kernel signature [Sun-Ovsjanikov-Guibas'09]

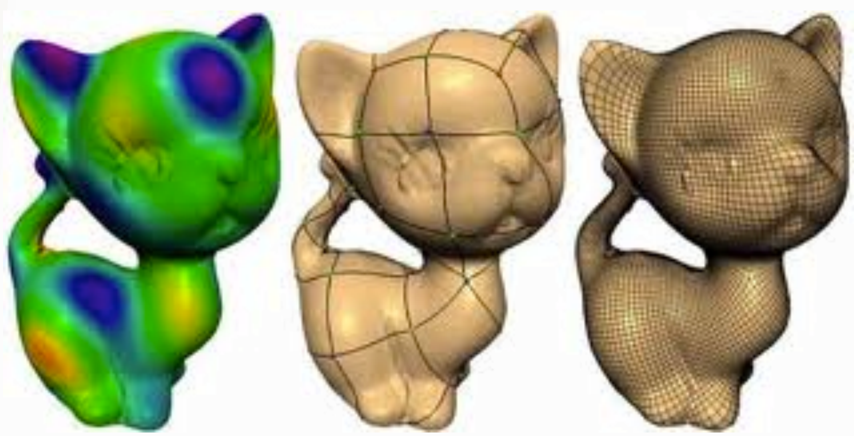
Relation between LB Eigen-system and surface geometry

- Nodal curves $\phi_k^{-1}(0)$ are smooth curves on $\mathcal{M}$ [S.Y.Cheng, 1976] , the connected components of $\mathcal{M} - \phi_k^{-1}(0)$ is called the k-th nodal number, which is between 2 and k [Courant nodal domain theorem]. The LB nodal counts can be used as complementary of LB eigenvalues. [Gnutzmann-Karageorge-Smilansky 2005, Lai-Shi-Toga-Chan'09]

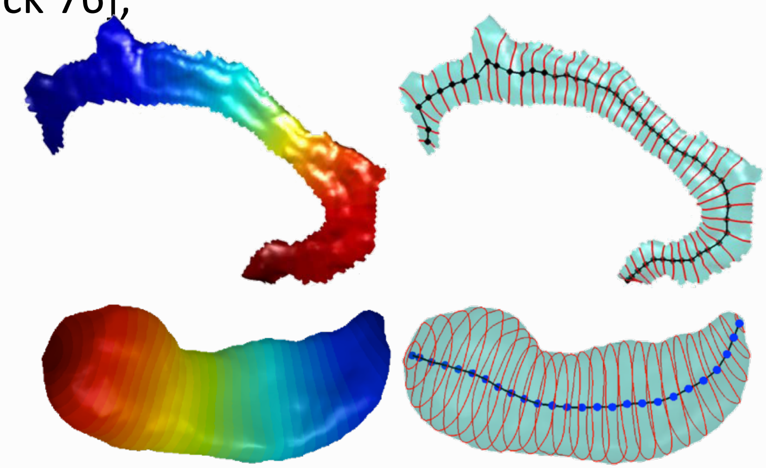




- LB eigenfunctions are Morse functions [Uhlenbeck'76],

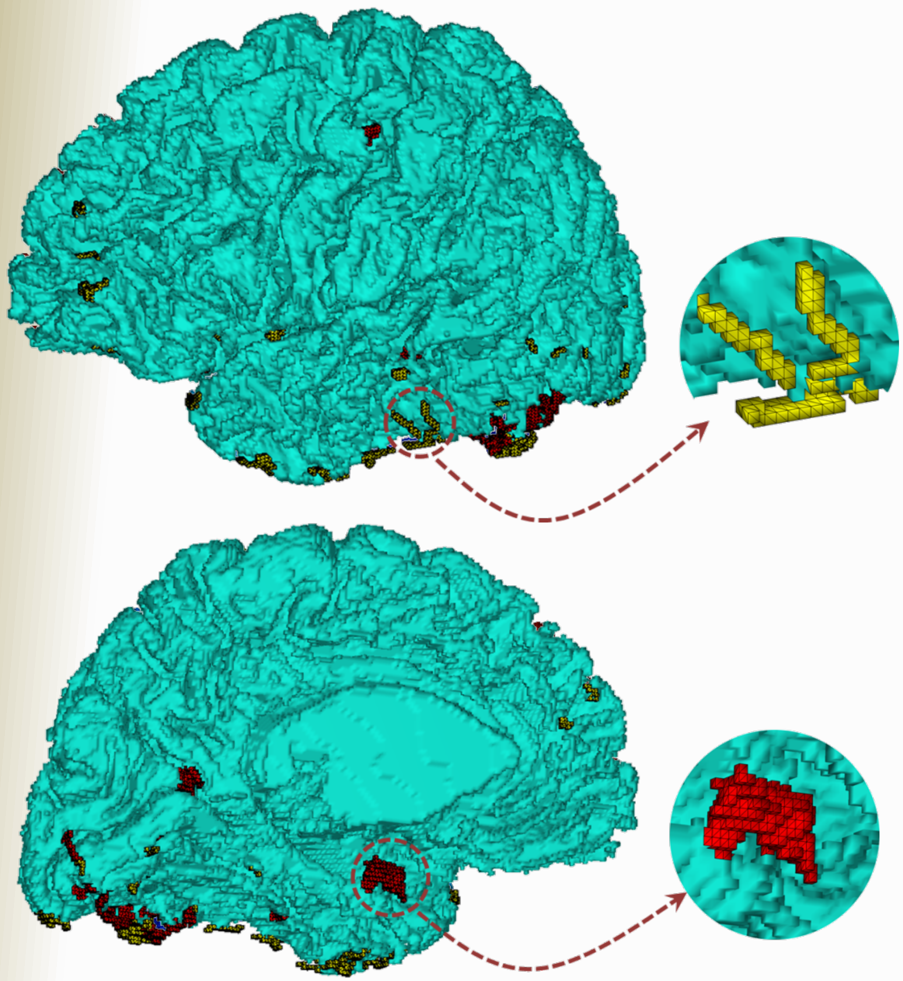


Morse-Smale complex. [Quadrangulation, Dong et al.'06]

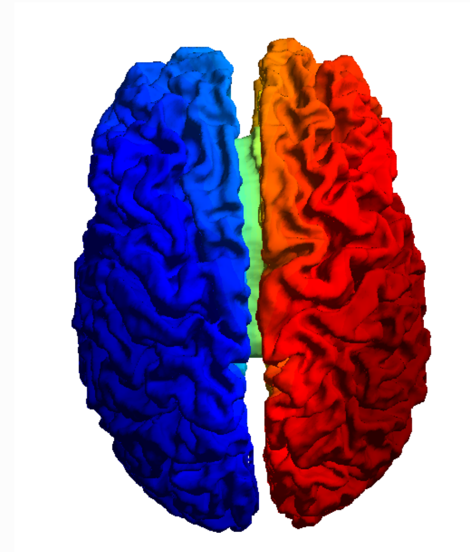


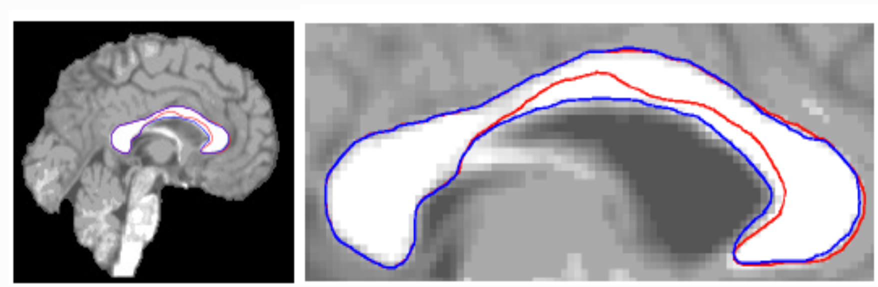
Skeleton construction using Reeb graphs [Shi-Lai-Krishna-Sicotte-Dinov-Toga,08]

Relation between LB Eigen-system and surface geometry



Unified Analysis of Geometric and Topological Outliers for Cortical Surface Reconstruction. [Shi-Lai-Toga]





Automated Corpus Callosum Extraction. [Shi-Lai-Toga'11]

LB Eigen-system and registration for near isometric surfaces

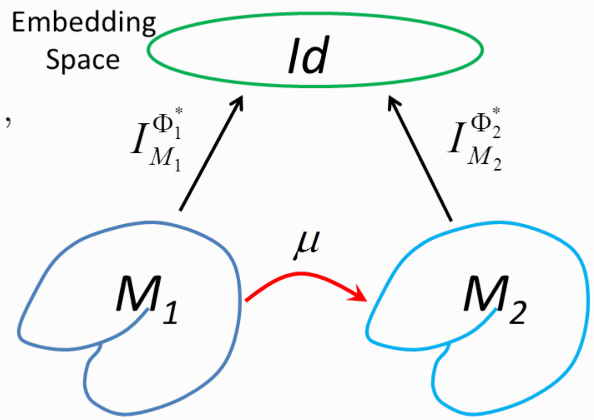
- A Riemannian manifold can be uniquely determined by its LB eigenvalues + eigenfunctions

- Heat kernel embedding [Berard-Besson-Gallot'94]: $I_t^{\Phi}(x) = \sqrt{Vol(\mathcal{M})}\{e^{-\lambda_j t/2}\phi_j(x)\}_{j\geq 1}$.
- Scale-invariant GPS embedding [Rustomov'07]: $I_{\mathcal{M}}^{\Phi}(x) = \left\{\frac{\phi_k(x)}{\lambda_j}\right\}_{j\geq 1}$

We define $d_{\Phi_1}^{\Phi_2}(x,\mathcal{M}_2) = \inf_{y\in M'} ||I_{\mathcal{M}_1}^{\Phi_1}(x) - I_{\mathcal{M}_2}^{\Phi_2}(y)||_2$.

$$d_{\Phi_1}^{\Phi_2}(\mathcal{M}_1,\mathcal{M}_2) = max\left\{\int_{\mathcal{M}_1} d_{\Phi_1}^{\Phi_2}(x,\mathcal{M}_2)\mathrm{d}vol_{\mathcal{M}_1}(x)\,,\int_{\mathcal{M}_2} d_{\Phi_1}^{\Phi_2}(\mathcal{M}_1,y)\mathrm{d}vol_{\mathcal{M}_2}(y)\right\},$$

$$d(\mathcal{M}_1,\mathcal{M}_2) = max\left\{\sup_{\Phi_1\in\mathcal{B}(\mathcal{M}_1)}\inf_{\Phi_2\in\mathcal{B}(\mathcal{M}_2)}\int_{\mathcal{M}} d_{\Phi_1}^{\Phi_2}(x,\mathcal{M}_2)\mathrm{d}vol_{\mathcal{M}}(x)\,,\right.$$
$$\left.\sup_{\Phi_2\in\mathcal{B}(\mathcal{M}_2)}\inf_{\Phi_1\in\mathcal{B}(\mathcal{M})}\int_{\mathcal{M}_2} d_{\Phi_1}^{\Phi_2}(\mathcal{M},y)\mathrm{d}vol_{\mathcal{M}_2}(y)\right\}.$$

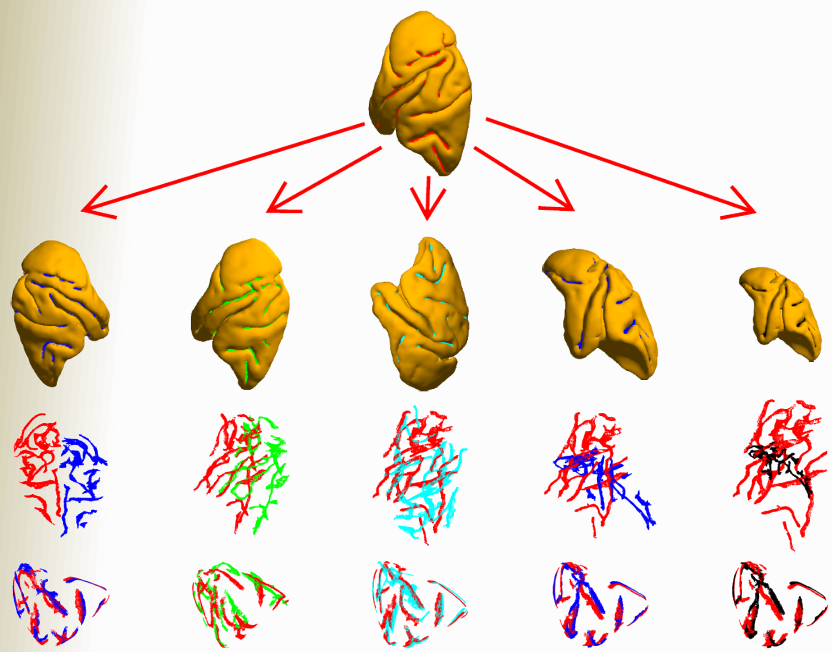


(Lai-Shi-Chan'10)

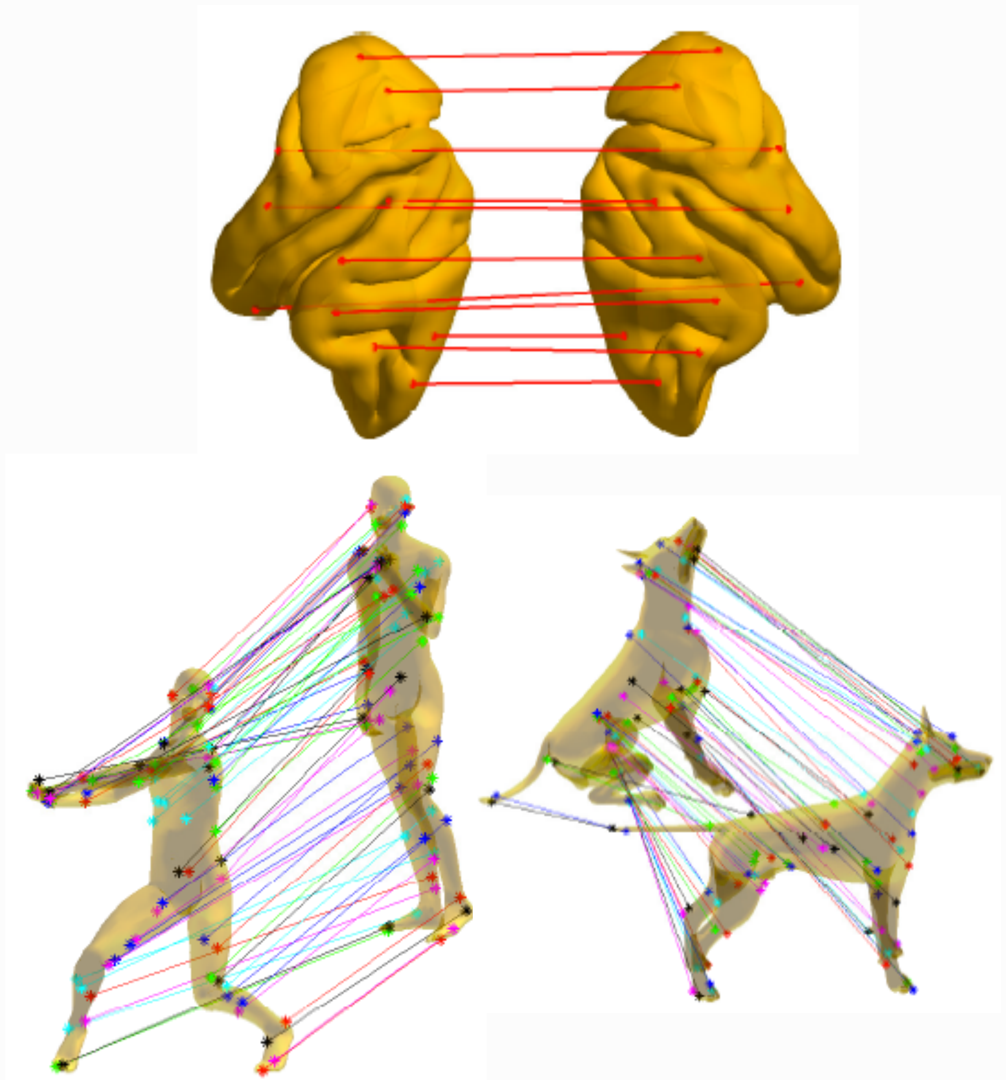
1. *(non-negativity and symmetry)* $d(\mathcal{M}_1,\mathcal{M}_2) \geqslant 0$ *and* $d(\mathcal{M}_1,\mathcal{M}_2) = d(\mathcal{M}_2,\mathcal{M}_1)$;
2. *(triangle inequality)* $d(\mathcal{M}_1,\mathcal{M}_2) \leqslant d(\mathcal{M}_1,N) + d(N,\mathcal{M}_2)$;
3. *(identity of indiscernibles)* $d(\mathcal{M}_1,\mathcal{M}_2) = 0 \Longleftrightarrow \mathcal{M}_1$ *isometric to* $\mathcal{M}_2$. *(*$\Longleftarrow$ *easy part,* $\Longrightarrow$ *hard part).*

Local pattern recognition/identification

Registration between near isometric surfaces



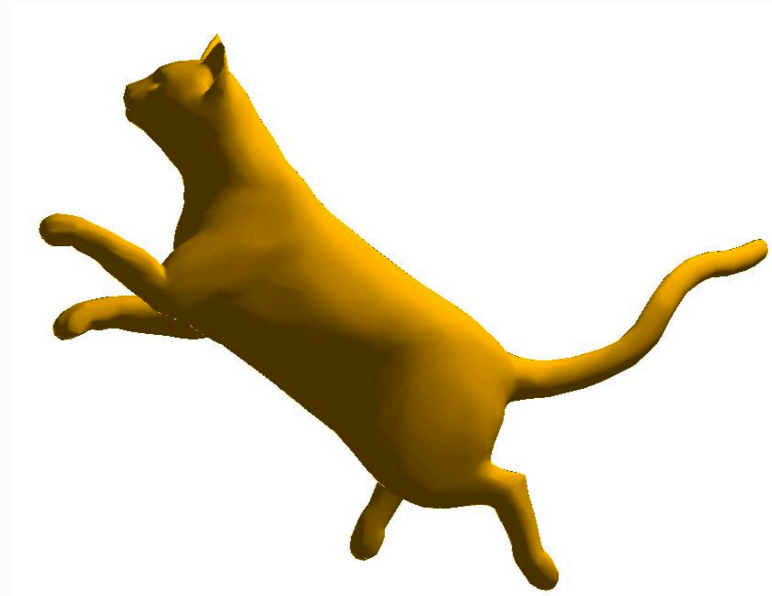
Sulci regions identification [Lai-Shi-Toga-Chan]



[Lai-Shi-Toga-Chan, Bronstein-Kimmel et.al]

Registration between non-isometric surfaces

Correspondence?

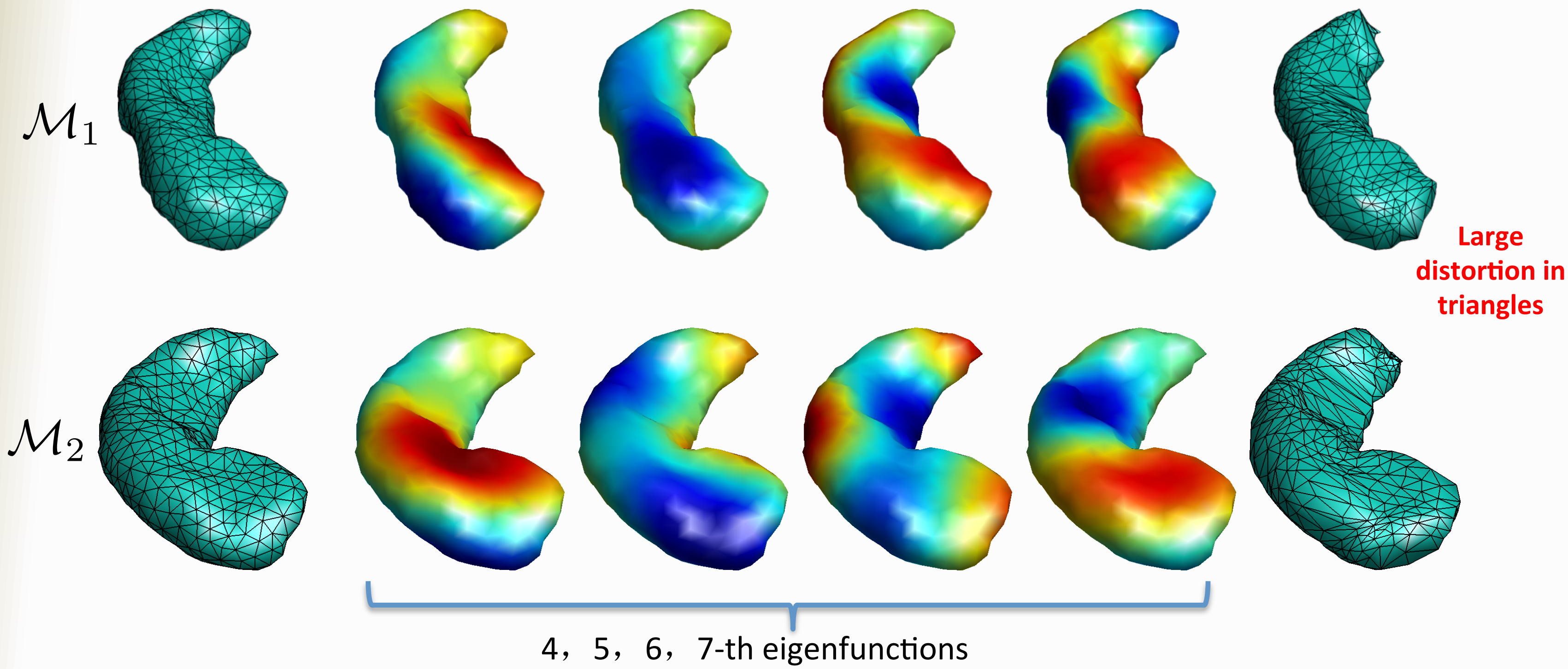


Challenges:

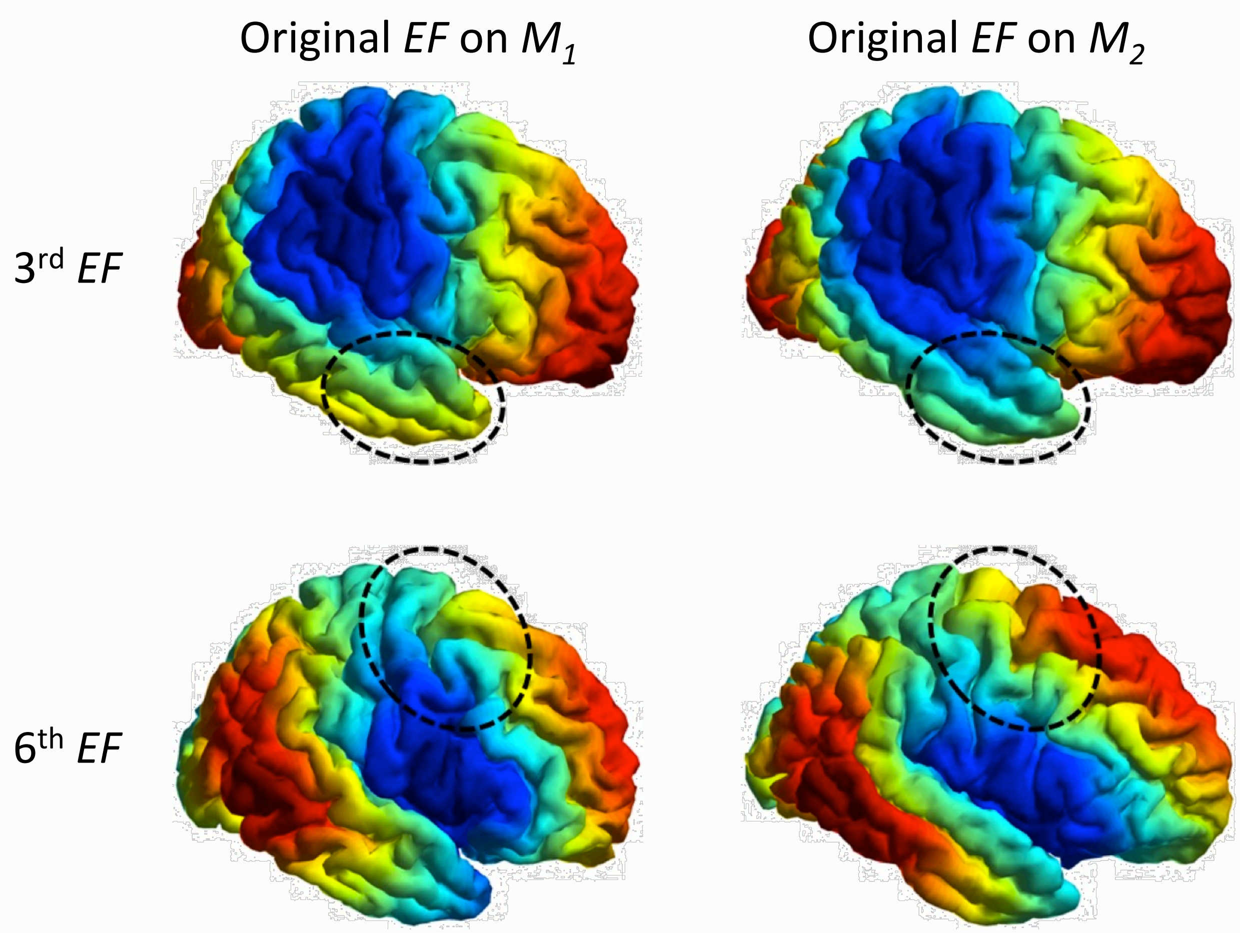
- Non-rigid
- Non-isometric

Non-isometric shape differences

- Non-isometric differences
  - Distance in the embedding space
  - Leads to artifact in surface mapping

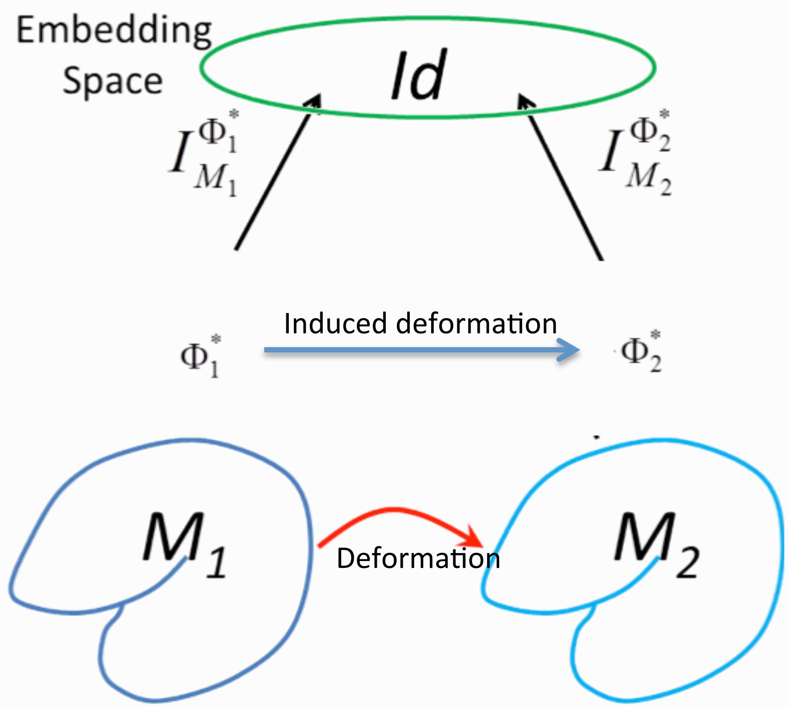


Cortical Examples



Non-isometric shape differences

- Strategy: deform $\mathcal{M}_1$ to $\mathcal{M}_2$



Challenges:

- How to measure the deformation?
- No correspondence

Conformal deformation

- Surface deformation
  - Extrinsic deformation: deform surfaces in the ambient space. e.g. mean curvature flow
  - Intrinsic deformation: deform surfaces intrinsically. No need to use ambient space at all. e.g. metric flows such as the Ricci flow.

- The shape space for genus-0 surfaces
  - All genus-0 surfaces are conformally equivalent.
  - $$\begin{aligned} &\{\text{all genus-0 surfaces}\} \\ \Longleftrightarrow\ &\{\text{fix a surface } \mathcal{M} \text{ with all different metric}\} \\ \Longleftrightarrow\ &\{(\mathcal{M}, g_0\omega) \mid \omega : \mathcal{M} \to \mathbb{R}^+\} \end{aligned}$$
  - Given a surface $(\mathcal{M}, g_0)$, we call a new metric $\hat{g}$ a conformal deformation of g_0, if there is $\omega : \mathcal{M} \to R^+$, s.t. $\hat{g} = \omega g_0$

LB eigenproblem on surfaces with conformal deformation

- The Laplace-Beltrami eigenproblem under the conformal deformation is

$$\Delta_{(\mathcal{M},\omega g_0)}\phi_k = -\lambda_k\phi_k \Longleftrightarrow \Delta_{(\mathcal{M},g_0)}\phi_k = -\omega\lambda_k\phi_k$$

[Proof:] Note that $|\omega g_0| = \omega^2 g_0$ and $(\omega g_0)^{ij} = \omega^{-1} g_0^{ij}$.

$$\begin{aligned}\Delta_{(\mathcal{M},\omega g_0)} f &= \frac{1}{\sqrt{|\omega g_0|}} \sum_{i=1,2} \partial_i \sqrt{|\omega g_0|} \sum_{j=1,2} (\omega g_0)^{ij} \partial_j f. \\ &= \frac{1}{\omega\sqrt{|g_0|}} \sum_{i=1,2} \partial_i \sqrt{|g_0|} \sum_{j=1,2} (g_0)^{ij} \partial_j f. \\ &= \frac{1}{\omega} \Delta_{(\mathcal{M},g_0)} f\end{aligned}$$

Weighted LB eigenproblem on the original surface $\Longleftrightarrow$ LB eigenproblem on conformal deformed surfaces

Non-isometric shape differences: Surface mapping with conformal deformation

- Key: All genus zero surfaces are conformally equivalent.
- Objective: Deform $(\mathcal{M}_1, g_1)$ to $(\mathcal{M}_2, g_2)$ via conformal deformation.
  - Challenge: Unknown mapping and unknown conformal deformation
  - Strategy: Simultaneously find mapping and unknown conformal deformation by iterative methods via certain energy minimization.
- Consider the following variational problem [Shi-Lai-Toga]:

$$(\omega_1^*, \omega_2^*) = arg \min_{\omega_1, \omega_2} \frac{1}{S_1} \int_{\mathcal{M}_1} \left(d_1^{\omega_1}\right)^2 (x)\, \mathrm{d}\mathcal{M}_1 + \frac{1}{S_2} \int_{\mathcal{M}_2} \left(d_2^{\omega_2}\right)^2 (y)\, \mathrm{d}\mathcal{M}_2 + \xi \sum_{i=1,2} \int_{\mathcal{M}_i} \|\nabla_{\mathcal{M}_i} \omega_i\|^2 \mathrm{d}\mathcal{M}_i$$

- Denote the conformal deformation of $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ by $\mathcal{M}_i^{\omega_i} = (\mathcal{M}_i, \omega_i g_i)(i = 1, 2)$.
- Write the optimal embedding bases of $(\mathcal{M}_1, \omega_1 g_1)$ and $(\mathcal{M}_2, \omega_2 g_2)$ as $\Phi_1^{\omega_1} = \{\lambda_{1,n}^{\omega_1}, \phi_{1,n}^{\omega_1}\}_{n=1}^{\infty}$ and $\Phi_2^{\omega_2} = \{\lambda_{2,n}^{\omega_2}, \phi_{2,n}^{\omega_2}\}_{n=1}^{\infty}$, respectively.
- Define: $d_1^{\omega_1}(x) = \inf_{y \in \mathcal{M}_2} ||I_{\mathcal{M}_1^{\omega_1}}^{\Phi_1^{\omega^1}}(x) - I_{\mathcal{M}_2^{\omega_2}}^{\Phi_2^{\omega^2}}(y)||_2 \ , \ \forall \ \ x \in \mathcal{M}_1$

$$d_2^{\omega_2}(y) = \inf_{x \in \mathcal{M}_1} ||I_{\mathcal{M}_1^{\omega_1}}^{\Phi_1^{\omega^1}}(x) - I_{\mathcal{M}_2^{\omega_2}}^{\Phi_2^{\omega^2}}(y)||_2 \ , \ \forall \ \ y \in \mathcal{M}_2,$$

Numerical computation

- Weak form

$$\int_{\mathcal{M}} \langle \nabla_{\mathcal{M}} \phi \nabla_{\mathcal{M}} \rangle \eta \, \mathrm{d}\mathcal{M} = \lambda \int_{\mathcal{M}} \omega \phi \eta \, \mathrm{d}\mathcal{M}, \quad \forall\, \eta \in C^{\infty}(\mathcal{M})$$

- Use the barycentric coordinate function e_i as the basis and test function

$$\omega = \sum_i \omega_i e_i, \quad f = \sum_i \beta_i e_i$$

- Matrix form $\quad Q\beta = \lambda \bar{U}(\omega)\beta$

where

$$Q_{ik} = \int_{\mathcal{M}} \langle \nabla_{\mathcal{M}} e_i, \nabla_{\mathcal{M}} e_k \rangle$$

$$\bar{U}_{ik}(\omega) = \sum_j \omega_j U_{ijk} = \sum_j \omega_j \int_{\mathcal{M}} e_i e_j e_k$$

- By solving the matrix eigenvalue problem, we have the embedding $\Phi^{\omega}_{\mathcal{M}}$ under the conformal deformation $\hat{g} = \omega g$

Variations of LB eigensystems via conformal deformation

Let (λ, ψ) be a simple eigenpair of $-\Delta_{\mathcal{M},\omega g}$. The variation of λ with respect to a perturbation of the conformal function ω is given by

$$\left(\frac{\delta\lambda}{\delta\omega},\ \delta\omega\right)_{\omega g} = -\lambda \frac{\left(\omega^{-1}\psi^2\ ,\delta\omega\right)_{\omega g}}{(\psi,\psi)_{\omega g}}$$

- Derivative of eigenvalues and eigenfunctions of $Qf_n = \lambda_n \bar{U}(\omega) f_n$ w.r.t. ω

$$Q\frac{\partial f_n}{\partial \omega_j} = \frac{\partial \lambda_n}{\partial \omega_j}\bar{U} f_n + \lambda_n \frac{\partial \bar{U}}{\partial \omega_j} f_n + \lambda_n \bar{U} \frac{\partial f_n}{\partial \omega_j}$$

$$f_n^T \bar{U} f_n = 1$$

- Since $f_n^T \bar{U} f_n = 1 \text{ and } f_n^T(Q - \lambda_n \bar{U}) = 0$ [Nelson'76]

$$\frac{\partial \lambda_n}{\partial \omega_j} = -\lambda_n f_n^T \frac{\partial \bar{U}}{\partial \omega_j} f_n \quad \text{and} \quad (Q - \lambda_n \bar{U})\frac{\partial f_n}{\partial \omega_j} = \frac{\partial \lambda_n}{\partial \omega_j}\bar{U} f_n + \lambda_n \frac{\partial \bar{U}}{\partial \omega_j} f_n$$

Surface Mapping in the Embedding Space

Discretization

- The energy discretization

$$E(\omega_1,\omega_2)=\sum_{n=1}^{N}\left(\frac{1}{S_1}\left(\frac{f_{1,n}}{\sqrt{\lambda_{1,n}}}-\frac{f_{2,n}(\mathbf{u}_1)}{\sqrt{\lambda_{2,n}}}\right)^T U_1\left(\frac{f_{1,n}}{\sqrt{\lambda_{1,n}}}-\frac{f_{2,n}(\mathbf{u}_1)}{\sqrt{\lambda_{2,n}}}\right)\right.$$
$$\left.+\frac{1}{S_2}\left(\frac{f_{2,n}}{\sqrt{\lambda_{2,n}}}-\frac{f_{1,n}(\mathbf{u}_2)}{\sqrt{\lambda_{1,n}}}\right)^T U_2\left(\frac{f_{2,n}}{\sqrt{\lambda_{2,n}}}-\frac{f_{1,n}(\mathbf{u}_2)}{\sqrt{\lambda_{1,n}}}\right)\right)+\xi(\omega_1^T Q_1\omega_1+\omega_2^T Q_2\omega_2)$$

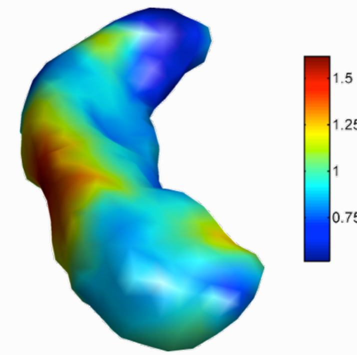
- Update weights iteratively

$$\frac{\partial E}{\partial\omega_1}=2\sum_{n=1}^{N}\left[\frac{1}{S_1}\left(\frac{1}{\sqrt{\lambda_{1,n}}}\frac{\partial f_{1,n}}{\partial\omega_1}-\frac{\partial\lambda_{1,n}}{\partial\omega_1}\frac{(f_{1,n})^T}{2\sqrt[3/2]{\lambda_{1,n}}}\right)U_1\left(\frac{f_{1,n}}{\sqrt{\lambda_{1,n}}}-\frac{Af_{2,n}}{\sqrt{\lambda_{2,n}}}\right)\right.$$
$$\left.-\frac{1}{S_2}\left(\frac{\partial f_{1,n}}{\partial\omega_1}\frac{B^T}{\sqrt{\lambda_{1,n}}}-\frac{\partial\lambda_{1,n}}{\partial\omega_1}\frac{(Bf_{1,n})^T}{2\sqrt[3/2]{\lambda_{1,n}}}\right)U_2\left(\frac{f_{2,n}}{\sqrt{\lambda_{2,n}}}-\frac{Bf_{1,n}}{\sqrt{\lambda_{1,n}}}\right)\right]+2\xi Q_1\omega_1$$

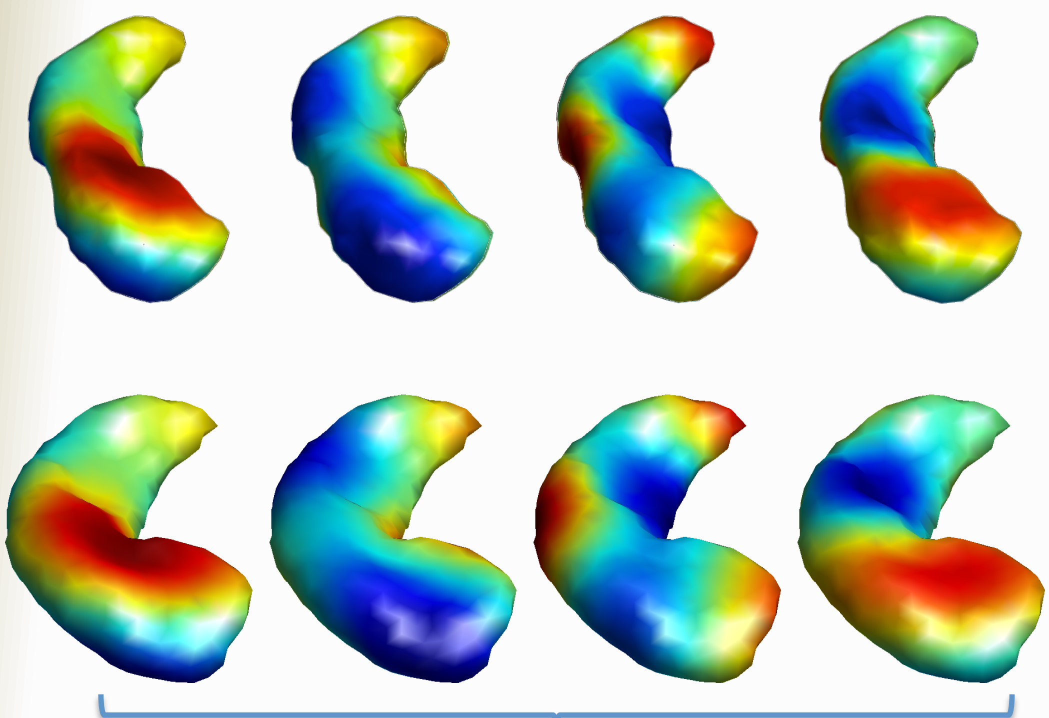
$$\frac{\partial E}{\partial\omega_2}=2\sum_{n=1}^{N}\left[\frac{1}{S_2}\left(\frac{1}{\sqrt{\lambda_{2,n}}}\frac{\partial f_{2,n}}{\partial\omega_2}-\frac{\partial\lambda_{2,n}}{\partial\omega_2}\frac{(f_{2,n})^T}{2\sqrt[3/2]{\lambda_{2,n}}}\right)U_2\left(\frac{f_{2,n}}{\sqrt{\lambda_{2,n}}}-\frac{Bf_{1,n}}{\sqrt{\lambda_{1,n}}}\right)\right.$$
$$\left.-\frac{1}{S_1}\left(\frac{\partial f_{2,n}}{\partial\omega_2}\frac{A^T}{\sqrt{\lambda_{2,n}}}-\frac{\partial\lambda_{2,n}}{\partial\omega_2}\frac{(Af_{2,n})^T}{2\sqrt[3/2]{\lambda_{2,n}}}\right)U_1\left(\frac{f_{1,n}}{\sqrt{\lambda_{1,n}}}-\frac{Af_{2,n}}{\sqrt{\lambda_{2,n}}}\right)\right]+2\xi Q_2\omega_2$$

Hippocampal Mapping Results

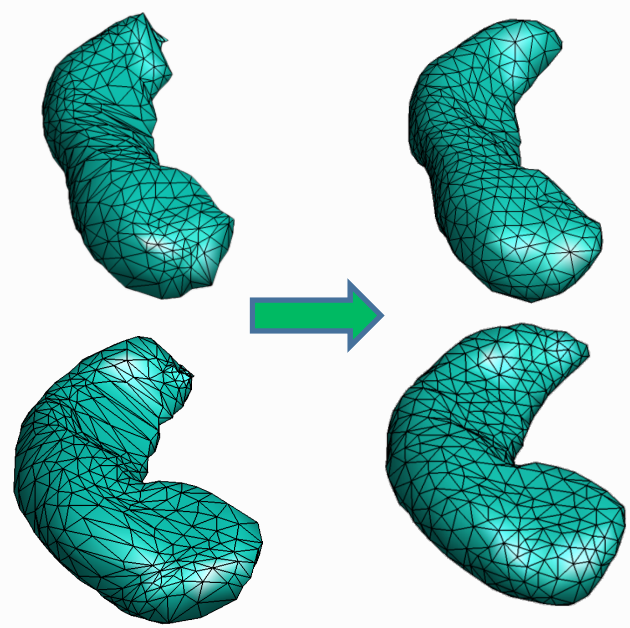
- Two hippocampal surfaces
  - Use 30 eigenfunctions in constructing the embedding space
  - Start with constant weights
- The weight of the source mesh are updated iteratively to compensate for the non isometric differences



Resulting conf. deform.



4，5，6，7-th eigenfunctions after conf. deform

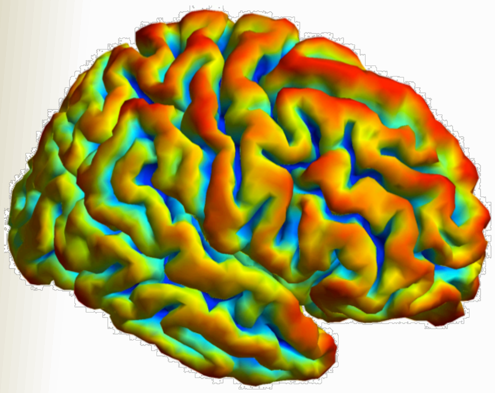


Before: Large distortion

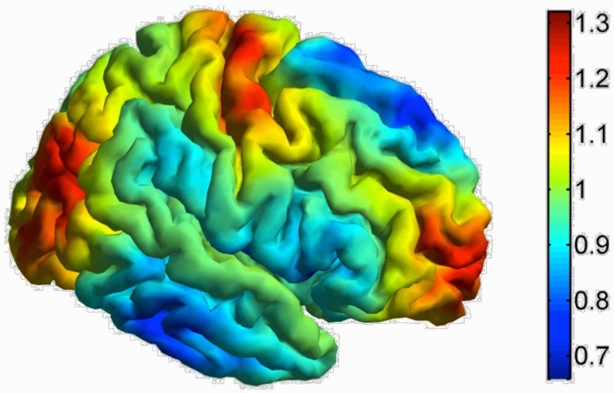
After: high quality map

Cortical mapping example

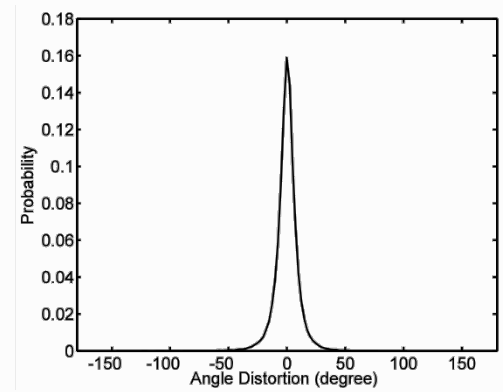
Source Surface M_1



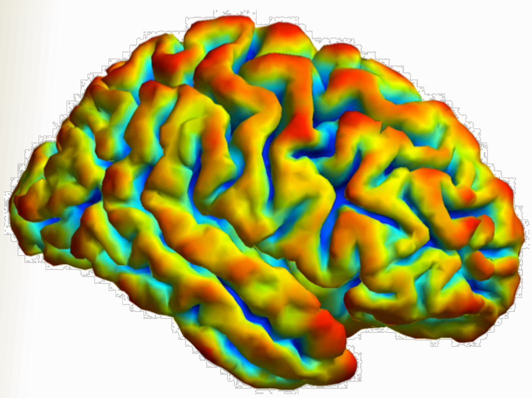
Optimized weight $\boldsymbol{w}$



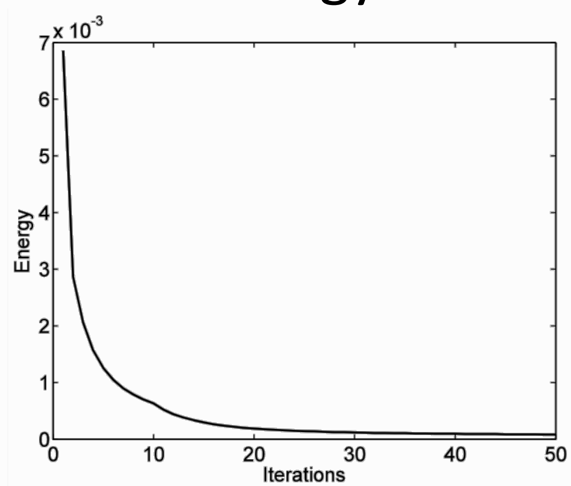
Angle Distortion



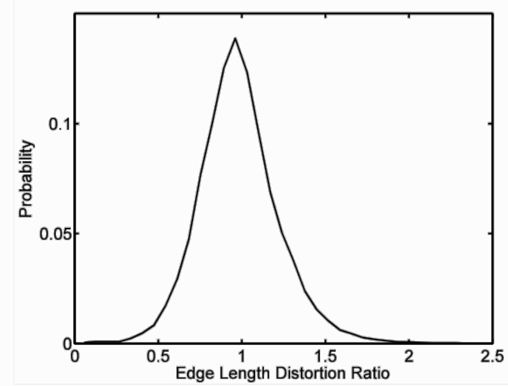
Target Surface M_2



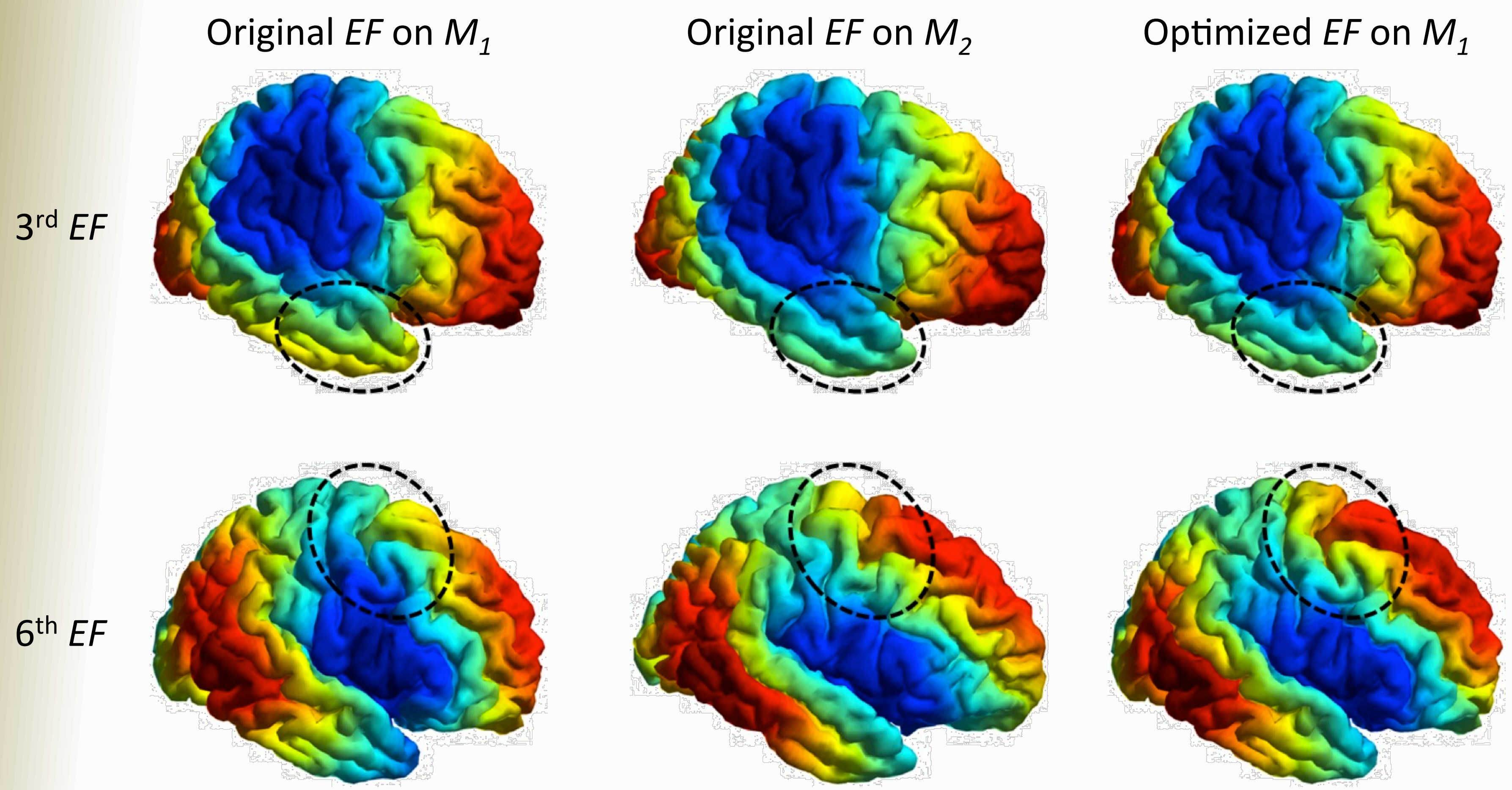
Energy



Metric Distortion



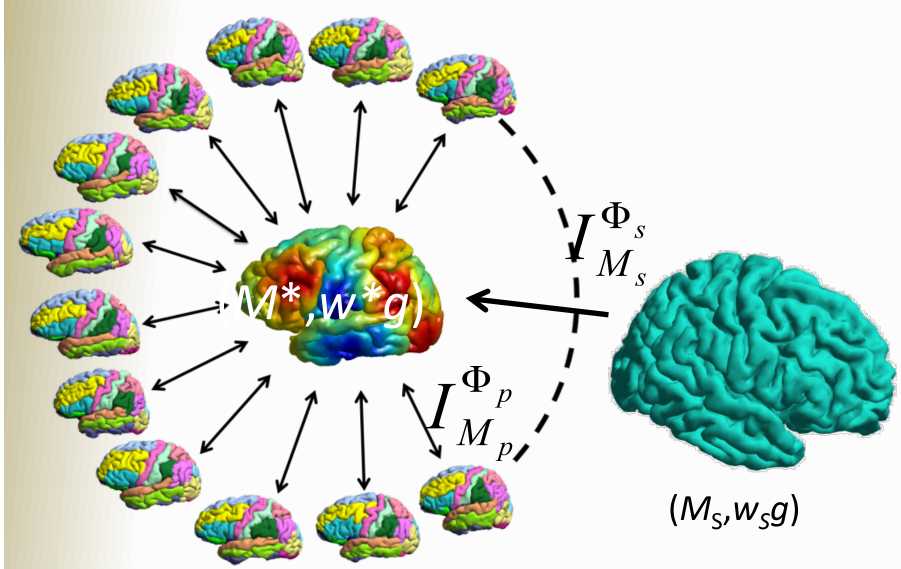
Optimized Embedding

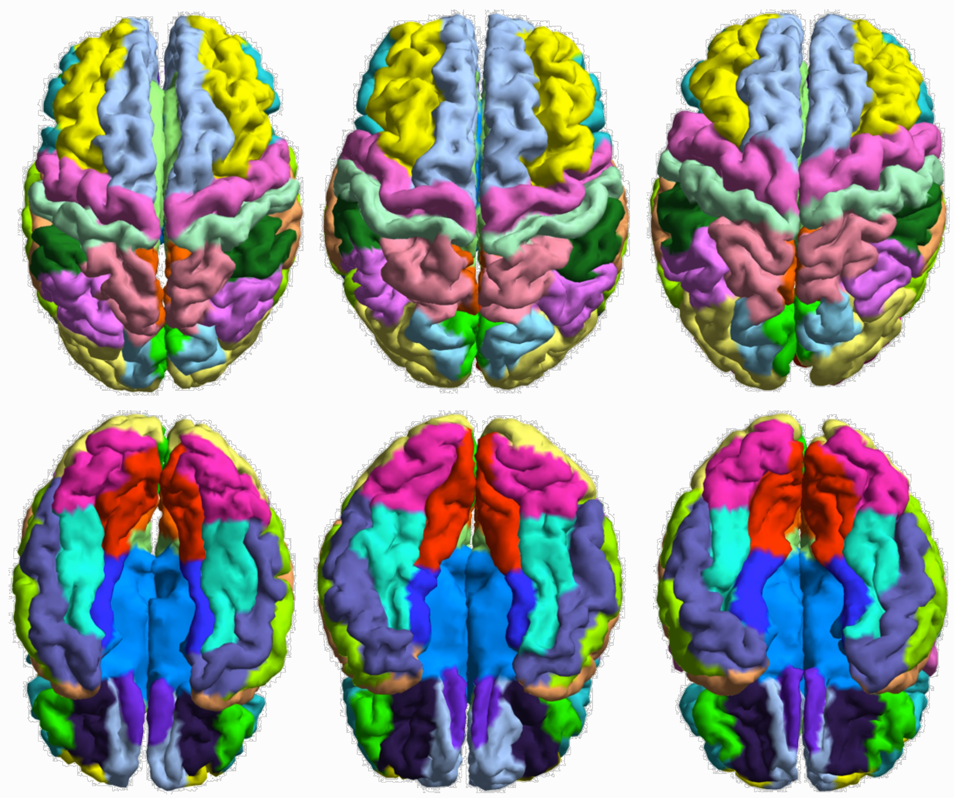


Group-wise atlas construction

- Given a set of annotated surfaces $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_p$, estimate a group-wise atlas $(\mathcal{M}^*, \omega^* g)$ for cortical label fusion
- Variational formula

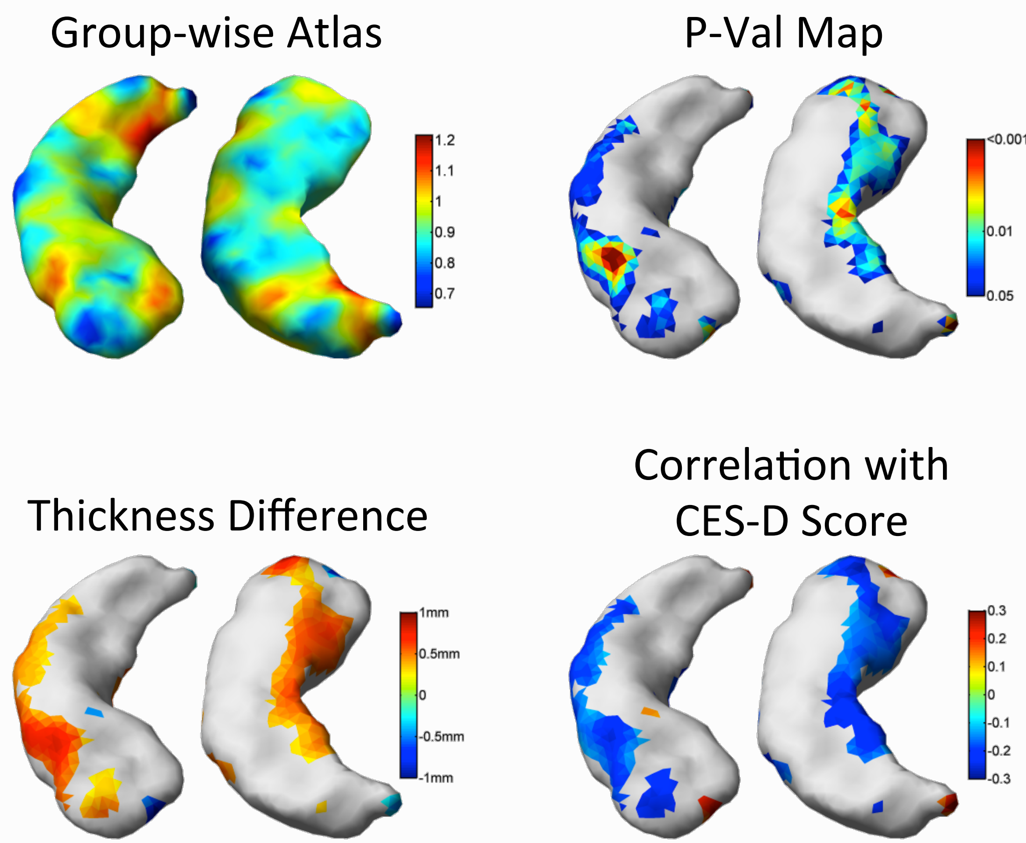
$$\arg\min_{\omega} \sum_{i=1}^{p} \int_{\mathcal{M}^*} \left(d_{\Phi^*}^{\Phi_i}(x, \mathcal{M}_i) \right)^2 d\mathcal{M}^* + \sum_{i=1}^{p} \int_{\mathcal{M}_p} \left(d_{\Phi_i}^{\Phi^*}(\mathcal{M}^*, x) \right)^2 d\mathcal{M}_p$$





Clinical Applications of Hippocampal Mapping

- Population study: hippocampal atrophy in multiple sclerosis (MS) patients with depression
  - 109 female patient split into two groups with the CES-D scale: low depression (CES-D≤20) and high depression (CES-D>20)
  - Statistically significant group differences were localized on hippocampus (P=0.019)
  - Correlates well with clinical measure of depression

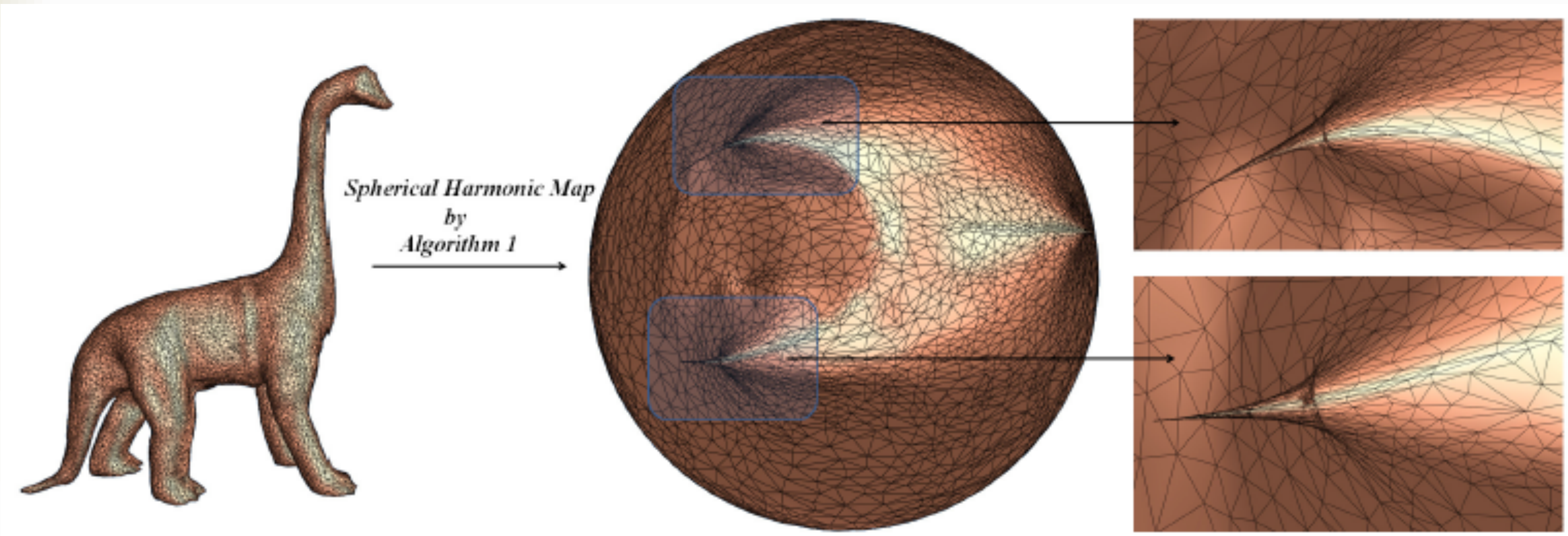


Folding free global conformal mapping [Lai-Wen-Yin-Gu-Lui]

- The variational formula for harmonic maps of genus-0 surfaces.

$$\min_{\vec{F}=(f_1,f_2,f_3)} \quad \mathcal{E}(\vec{F}) = \frac{1}{2}\int_{\mathcal{M}} \sum_{i=1}^{3} \|\nabla_{\mathcal{M}} f_i\|^2 \mathrm{d}\mathcal{M}, \quad s.t. \quad \|\vec{F}(x)\|^2 = 1, \quad \forall x \in \mathcal{M}.$$

  - Slow convergence of gradient projection method.
  - Possible foldings for surfaces with long and sharp features.



- Our new algorithms.
  - Speed up the computation efficiency tremendously using curvilinear search method [Wen-Glodfarb-Yin] or SOC [Lai-Osher].
  - Can obtain folding-free maps using LB eigensystem for conformal deformed surfaces.

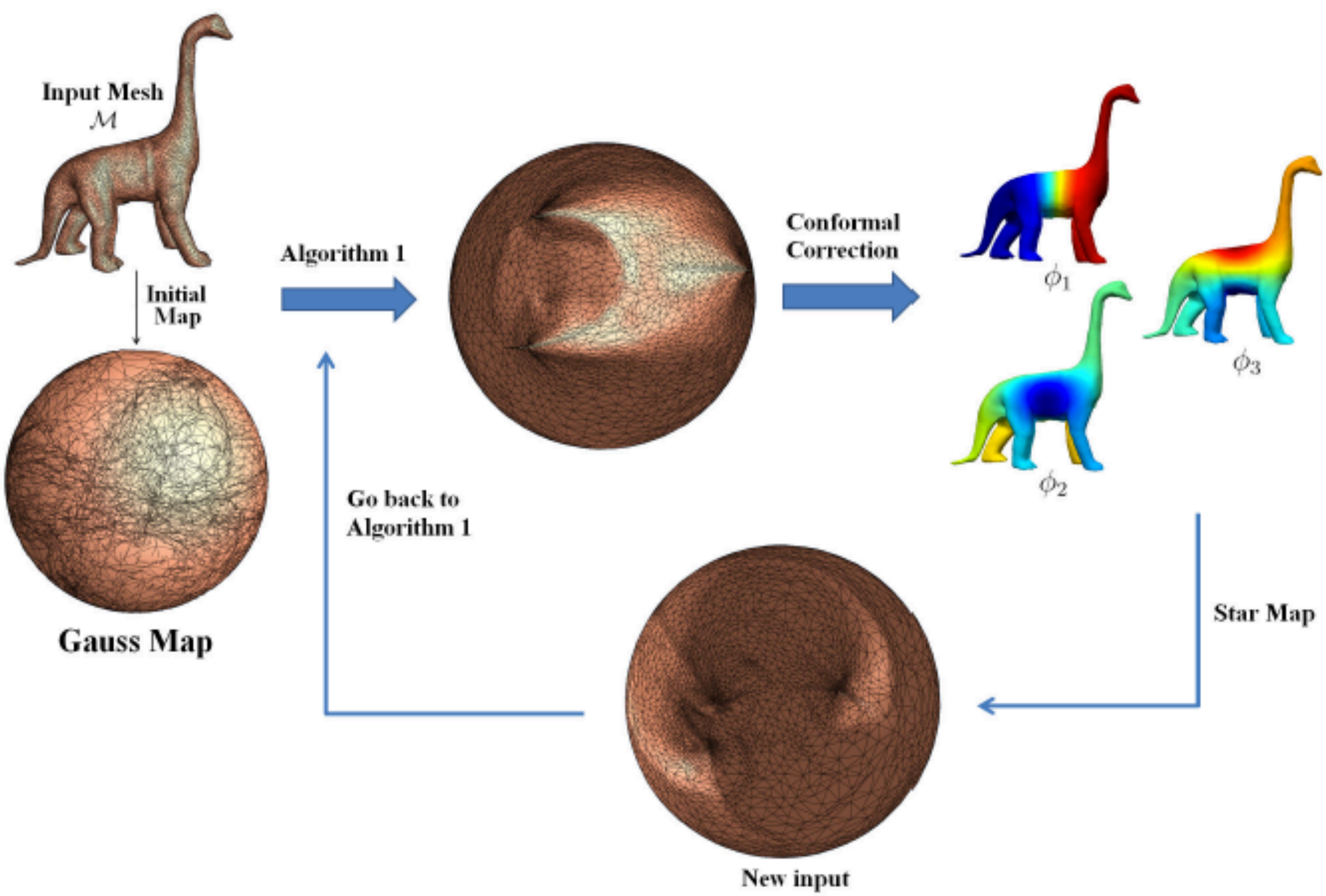
Folding free global conformal mapping [Lai-Wen-Yin-Gu-Lui]

Algorithm 2: Folding removal by weighted LB Eigen-projection

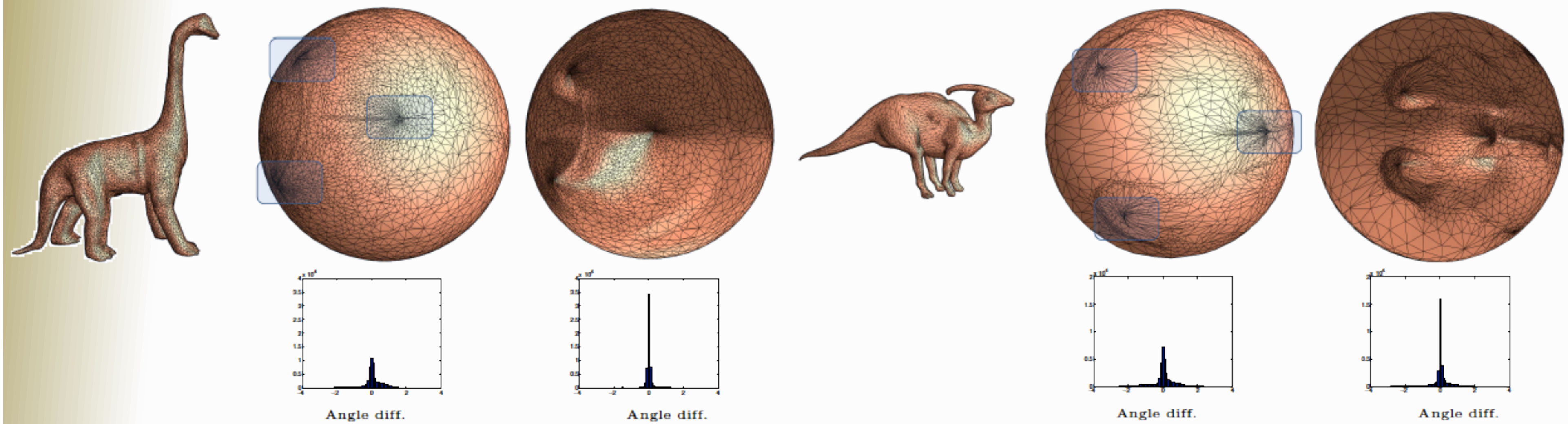
1. Compute a map $\mathbf{F}_0 : \mathcal{M} \to S^2$ using Algorithm 1.
   Compute the corresponding e^{2u_0} using the approximation formula (14).

Iterate the following steps starting from $k = 1$.

2. Given the map $\mathbf{F}_{k-1}$ and conformal factor $e^{2u_{k-1}}$, solve (20).
3. Construct a Star map using $\{\phi_1^{u_{k-1}}, \phi_2^{u_{k-1}}, \phi_3^{u_{k-1}}\}$.
4. Start Algorithm 1 for the Star map and obtain $\mathbf{F}_k$ and u_k.



Folding free global conformal mapping [Lai-Wen-Yin-Lui-Gu]



Surface	# of vertices	ϵ	The proposed Algorithm 2 **(Foldings Removed)**				Algorithm in [1] **(Foldings Remained)**	
			Iter 0 step	Iter 1 step	Iter 2 step	time(s)	# of iterations	time(s)
Dino	5524	1e-10	4480	1518	3652	91.11	2854	835.53
Dilo	9731	1e-10	3610	5000	4184	73.94	3106	513.56
Bird 1	950	1e-10	1054	4444	1038	2.64	716	19.29
Armadillo	16519	1e-10	4164	294	496	70.68	3355	1648.30

Table 3 Comparison between the proposed Algorithm 2 and the algorithm in [1].

Computing the conformal and topological spectra of Riemannian Surfaces (Ongoing project with Chiu-Yen Kao and Braxton Osting)

- The Conformal spectrum.

  Let $[g_0]$ be the conformal class of g_0 with fixed $vol(\mathcal{M}, g) = 1$. The k-th conformal LB eigenvalue of $(\mathcal{M}, g_0)$ is:

$$\lambda_k^C(\mathcal{M}, [g_0]) = \sup_{g \in [g_0]} \lambda_k(\mathcal{M}, g) = \sup\{\lambda_k(\mathcal{M}, g) vol(\mathcal{M}, g) \mid g \text{ conformal to } g_0\}$$

- The topological spectrum of a 2-dim closed surface with genus γ .

$$\lambda_k^T(\gamma) = \sup_{g \in G(\mathcal{M}_\gamma)} \lambda_k(\mathcal{M}_\gamma, g).$$

- In our work:

1. For a given Riemannian manifold $(\mathcal{M}, g_0)$, we develop a computational method for finding a metric $g \in [g_0]$ which attains the conformal spectrum.

2. We also develop a computational method for studying the topological spectrum for genus $\gamma = 0$ and $\gamma = 1$, which depends on a parameterization of moduli space.

3. We study the flat tori and embedded tori.

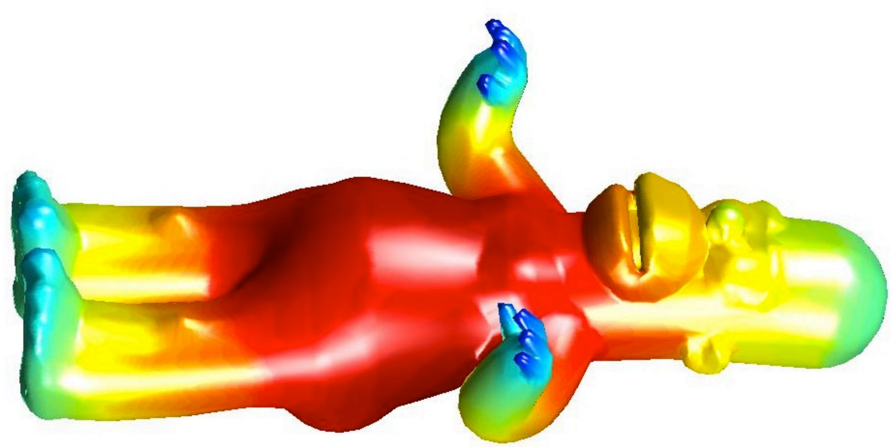
A new variational method for computing conformal factor [Kao-Lai-Osting]

- All genus-0 surfaces are conformal equivalent.
- The conformal factor for a genus-0 surface M satisfies the following highly nonlinear equation (Yamabe problem):

$$\Delta_{\mathcal{M}} u + \tilde{K} e^{2u} - K = 0$$

whose solution is not straightforward to obtain. ($\tilde{K} = 1$ for the unit sphere)

- The optimizer of $\lambda_1^C(\mathcal{M}, [g_0]) = \sup_{g \in [g_0]} \lambda_1(\mathcal{M}, g)$ is the canonical metric $\mathbb{S}^2$ [Hersch'70]



Conformal factor of Homer Simpson

Conclusion and future work

We consider the surface conformal deformation and study the Laplace-Beltrami eigenproblem on surfaces with conformal deformation

- Conformal deformation for registration between non isometric surfaces. We demonstrate its applications on surfaces from Brain image. It can be used to more general registration problems in computer graphics.

- Folding free global conformal maps.

- Computing conformal and topological spectrum

- A new variational method for solving Yamabe problem for genus-0 surfaces.

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Thanks! Question?

Rongjie Lai
University of California, Irvine
rongjiel@math.uci.edu