Applications of Laplace-Beltrami Spectrum via Conformal Deformation

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Motivation and Problems

Portable 3D scanner

Magnetic Resonance scanner
Laplace-Beltrami Eigen-system

A Laplace-Beltrami (LB) eigen-system of \((\mathcal{M}, g)\):

\[
\begin{align*}
\Delta_{\mathcal{M}} \phi_k &= -\lambda_k \phi_k, \quad k = 1, 2, \cdots, \\
\partial_{\nu} \phi_k(x) &= 0, \quad x \in \partial \mathcal{M}
\end{align*}
\]

- \(\Delta_{\mathcal{M}} \phi_k = \frac{1}{\sqrt{G}} \partial_{x^i} (\sqrt{G} g^{i\bar{j}} \partial_{x^\bar{j}} \phi_k)\) is the LB operator of \((\mathcal{M}, g)\).
- \(\sigma(\mathcal{M}, g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\}\) are eigenvalues of \(\Delta_{\mathcal{M}}\) on \((\mathcal{M}, g)\).
- The corresponding \(\phi_0, \phi_1, \phi_2, \cdots\) are called eigenfunctions.

Example: Fourier basis and spherical harmonics
Computation of Laplace-Beltrami Eigen-system

- Implicit approach: Level set method \( \mathcal{M} = f^{-1}(0) \subset \mathbb{R}^3 \)
  - J. Brandman’08, The first few LB eigenfunctions \( \leftrightarrow \) a PDE (the tangential component) + an ODE (normal component).
  - Gao-Lai-Shi’10 propose to approximate the LB eigenproblem on the narrow band \( \mathcal{M}_\delta = f^{-1}([-\delta, \delta]) \). If \( \frac{\pi^2}{4\delta^2} > \lambda_k(\mathcal{M}) \), then \( \lambda_i(\mathcal{M}_\delta) = \lambda_i(\mathcal{M}), \quad i = 0, \ldots, k \)

- Closest point method. [C. Macdonald-J. Brandman-S. Ruuth]

- Triangle mesh based method
  - Finite difference method to approximate the LB operator. [Taubin’00,Desbrun Meyer-Schroder-Bar’02, Xu’04], : \( \Delta_{\mathcal{M}} f(v_i) \approx \sum_{j \in N(i)} \omega_{ij} (f(v_j) - f(v_i)) \)
  - Finite element method based on the weak formula [M. Reuter’06, A.Qiu’06]

- Point cloud based method, PDEs on high dimensional manifolds.
  - Diffusion based method, only for Laplacian [Belkin-Sun-Wang,Coifman-Lafon]
  - Two systematic methods, can be applied for other PDEs. [Lai-Liang-Zhao]
Relation between LB Eigen-system and surface geometry

- Inverse spectral problem: Can one hear the shape of a drum? [Kac’66]
  - Heat trace asymptotic expansion:
    $$Z(t) = \int_{\mathcal{M}} K(t, x, x) \text{dv}(x) = \frac{1}{4\pi t} \sum_i c_i t^{i/2}$$
    \[c_0 = \text{area}(\mathcal{M}), \quad c_1 = -\frac{\sqrt{\pi}}{2} \text{length}(\partial \mathcal{M}), \quad c_2 = \frac{1}{3} \int_{\mathcal{M}} K - 1/6 \int_{\partial \mathcal{M}} J\]
    where $K(t, x, y) = \sum_i e^{-\lambda_i t} \phi_i(x)\phi_i(y)$, $K$ is the Gauss curvature of $\mathcal{M}$ and $J$ is the mean curvature of $\partial \mathcal{M}$ in $\mathcal{M}$. Moreover, if $\mathcal{M}$ is a closed surface with Euler number $\chi(\mathcal{M})$, then $c_2 = 2\chi(\mathcal{M})/3$. [H. Mckean, I. Singer’67]

- Isospectral surfaces. LB eigenvalues can not uniquely determine a surface. [Milnor,Sunada, Gordon-Webb-Wolpert et al.]

- Asymptotical behavior [Weyl, 1910]
  $$\lambda_k \sim \frac{4\pi^2 k^{2/d}}{(C_d \text{Vol}(\mathcal{M}))^{2/d}} \text{ as } k \to \infty, \text{ here } C_d = \text{vol}(d\text{-ball})$$

- Shape DNA[Reuter’06], Heat kernel signature [Sun-Ovsjanikov-Guibas’09]
Relation between LB Eigen-system and surface geometry

- Nodal curves \( \phi_k^{-1}(0) \) are smooth curves on \( \mathcal{M} \) [S.Y. Cheng, 1976], the connected components of \( \mathcal{M} - \phi_k^{-1}(0) \) is called the k-th nodal number, which is between 2 and k [Courant nodal domain theorem]. The LB nodal counts can be used as complementary of LB eigenvalues. [Gnutzmann-Karageorge-Smilansky 2005, Lai-Shi-TogaChan’09]

- LB eigenfunctions are Morse functions [Uhlenbeck’76],

Morse-Smale complex. [Quadrangulation, Dong et al.’06]

Skeleton construction using Reeb graphs [Shi-Lai-Krishna-Sicotte-Dinov-Toga, 08]
Relation between LB Eigen-system and surface geometry

Unified Analysis of Geometric and Topological Outliers for Cortical Surface Reconstruction. [Shi-Lai-Toga]

Automated Corpus Callosum Extraction. [Shi-Lai-Toga’11]
LB Eigen-system and registration for near isometric surfaces

- A Riemannian manifold can be uniquely determined by its LB eigenvalues + eigenfunctions
- Heat kernel embedding [Berard-Besson-Gallot’94]: \( I_t^\Phi(x) = \sqrt{Vol(M)} \{ e^{-\lambda_j t/2} \phi_j(x) \}_{j \geq 1} \).
- Scale-invariant GPS embedding [Rustomov’07]: \( I_{\mathcal{M}}^\Phi(x) = \left\{ \frac{\phi_k(x)}{\lambda_j} \right\}_{j \geq 1} \).

We define \( d_{\Phi_1}^{\Phi_2}(x, \mathcal{M}_2) = \inf_{y \in \mathcal{M}_2} \| I_{\mathcal{M}_1}^{\Phi_1}(x) - I_{\mathcal{M}_2}^{\Phi_2}(y) \|_2 \).

\[
d_{\Phi_1}(\mathcal{M}_1, \mathcal{M}_2) = \max \left\{ \int_{\mathcal{M}_1} d_{\Phi_1}^{\Phi_2}(x, \mathcal{M}_2) d\nu M_1(x), \int_{\mathcal{M}_2} d_{\Phi_1}^{\Phi_2}(\mathcal{M}_1, y) d\nu M_2(y) \right\}.
\]

\[
d(\mathcal{M}_1, \mathcal{M}_2) = \max \left\{ \sup_{\Phi_1 \in \mathcal{B}(\mathcal{M}_1)} \inf_{\Phi_2 \in \mathcal{B}(\mathcal{M}_2)} \int_{\mathcal{M}} d_{\Phi_1}^{\Phi_2}(x, \mathcal{M}_2) d\nu M(x), \right. \\
\left. \sup_{\Phi_2 \in \mathcal{B}(\mathcal{M}_2)} \inf_{\Phi_1 \in \mathcal{B}(\mathcal{M}_1)} \int_{\mathcal{M}_2} d_{\Phi_1}^{\Phi_2}(\mathcal{M}_1, y) d\nu M_2(y) \right\}.
\]

(Lai-Shi-Chan’10)

1. (non-negativity and symmetry) \( d(\mathcal{M}_1, \mathcal{M}_2) \geq 0 \) and \( d(\mathcal{M}_1, \mathcal{M}_2) = d(\mathcal{M}_2, \mathcal{M}_1) \);
2. (triangle inequality) \( d(\mathcal{M}_1, \mathcal{M}_2) \leq d(\mathcal{M}_1, N) + d(N, \mathcal{M}_2) \);
3. (identity of indiscernibles) \( d(\mathcal{M}_1, \mathcal{M}_2) = 0 \iff \mathcal{M}_1 \text{ isometric to } \mathcal{M}_2 \). (easy part, hard part).
Local pattern recognition/identification

Registration between near isometric surfaces

Sulci regions identification [Lai-Shi-Toga-Chan]

[Lai-Shi-Toga-Chan, Bronstein-Kimmel et.al]
Registration between non-isometric surfaces

Challenges:
- Non-rigid
- Non-isometric
Non-isometric shape differences

- Non-isometric differences
  - Distance in the embedding space
  - Leads to artifact in surface mapping

\[ M_1 \]

\[ M_2 \]

Large distortion in triangles

4, 5, 6, 7-th eigenfunctions
Cortical Examples

Original $EF$ on $M_1$

3rd $EF$

Original $EF$ on $M_2$

6th $EF$
Non-isometric shape differences

- **Strategy:** deform $\mathcal{M}_1$ to $\mathcal{M}_2$

Challenges:
- How to measure the deformation?
- No correspondence
Conformal deformation

- **Surface deformation**
  - Extrinsic deformation: deform surfaces in the ambient space. e.g. mean curvature flow
  - Intrinsic deformation: deform surfaces intrinsically. No need to use ambient space at all. e.g. metric flows such as the Ricci flow.

- **The shape space for genus-0 surfaces**
  - All genus-0 surfaces are conformally equivalent.
  - Given a surface \((M, g_0)\), we call a new metric \(\hat{g}\) a conformal deformation of \(g_0\), if there is \(\omega : M \rightarrow \mathbb{R}^+\), s.t. \(\hat{g} = \omega g_0\)
The Laplace-Beltrami eigenproblem under the conformal deformation is

\[ \Delta_{(\mathcal{M}, \omega g_0)} \phi_k = -\lambda_k \phi_k \quad \Longleftrightarrow \quad \Delta_{(\mathcal{M}, g_0)} \phi_k = -\omega \lambda_k \phi_k \]

[Proof:] Note that \( |\omega g_0| = \omega^2 g_0 \) and \( (\omega g_0)^{ij} = \omega^{-1} g_0^{ij} \).

\[
\begin{align*}
\Delta_{(\mathcal{M}, \omega g_0)} f &= \frac{1}{\sqrt{|\omega g_0|}} \sum_{i=1,2} \partial_i \sqrt{|\omega g_0|} \sum_{j=1,2} (\omega g_0)^{ij} \partial_j f. \\
&= \frac{1}{\omega \sqrt{|g_0|}} \sum_{i=1,2} \partial_i \sqrt{|g_0|} \sum_{j=1,2} (g_0)^{ij} \partial_j f. \\
&= \frac{1}{\omega} \Delta_{(\mathcal{M}, g_0)} f
\end{align*}
\]
Non-isometric shape differences: Surface mapping with conformal deformation

- **Key:** All genus zero surfaces are conformally equivalent.
- **Objective:** Deform \((M_1, g_1)\) to \((M_2, g_2)\) via conformal deformation.
  - **Challenge:** Unknown mapping and unknown conformal deformation
  - **Strategy:** Simultaneously find mapping and unknown conformal deformation by iterative methods via certain energy minimization.

- **Consider the following variational problem [Shi-Lai-Toga]:**

\[
(\omega^*_1, \omega^*_2) = \arg \min_{\omega_1, \omega_2} \frac{1}{S_1} \int_{M_1} (d_1^{\omega_1})^2(x) \, dM_1 + \frac{1}{S_2} \int_{M_2} (d_2^{\omega_2})^2(y) \, dM_2 + \xi \sum_{i=1,2} \int_{M_i} ||\nabla M_i \omega_i||^2 \, dM_i
\]

- Denote the conformal deformation of \((M_1, g_1)\) and \((M_2, g_2)\) by \(M_i^{\omega_i} = (M_i, \omega_i g_i)(i = 1, 2)\).

- Write the optimal embedding bases of \((M_1, \omega_1 g_1)\) and \((M_2, \omega_2 g_2)\) as \(\Phi_1^{\omega_1} = \{\lambda_1, n, \phi_1, n\}_{n=1}^\infty\) and \(\Phi_2^{\omega_2} = \{\lambda_2, n, \phi_2, n\}_{n=1}^\infty\), respectively.

- Define: 
  \[d_1^{\omega_1}(x) = \inf_{y \in M_2} ||I_{M_1}^{\Phi_1^{\omega_1}}(x) - I_{M_2}^{\Phi_2^{\omega_2}}(y)||_2, \quad \forall \ x \in M_1\]

  \[d_2^{\omega_2}(y) = \inf_{x \in M_1} ||I_{M_1}^{\Phi_1^{\omega_1}}(x) - I_{M_2}^{\Phi_2^{\omega_2}}(y)||_2, \quad \forall \ y \in M_2,\]
Numerical computation

- **Weak form**
  \[
  \int_{\mathcal{M}} \langle \nabla_{\mathcal{M}} \phi \nabla_{\mathcal{M}} \rangle \eta \, d\mathcal{M} = \lambda \int_{\mathcal{M}} \omega \phi \eta \, d\mathcal{M}, \quad \forall \eta \in C^\infty(\mathcal{M})
  \]

- **Use the barycentric coordinate function** $e_i$ **as the basis and test function**
  \[
  \omega = \sum_i \omega_i e_i, \quad f = \sum_i \beta_i e_i
  \]

- **Matrix form**
  \[
  Q \beta = \lambda \bar{U}(\omega) \beta
  \]
  where
  \[
  Q_{ik} = \int_{\mathcal{M}} \langle \nabla_{\mathcal{M}} e_i, \nabla_{\mathcal{M}} e_k \rangle
  \]
  \[
  \bar{U}_{ik}(\omega) = \sum_j \omega_j U_{ijk} = \sum_j \omega_j \int_{\mathcal{M}} e_i e_j e_k
  \]

- **By solving the matrix eigenvalue problem**, we have the embedding $\Phi_{\mathcal{M}}^\omega$
  under the conformal deformation $\hat{g} = \omega g$
Variations of LB eigensystems via conformal deformation

Let \((\lambda, \psi)\) be a simple eigenpair of \(-\Delta_{\mathcal{M}, \omega_g}\). The variation of \(\lambda\) with respect to a perturbation of the conformal function \(\omega\) is given by

\[
\begin{pmatrix}
\frac{\delta \lambda}{\delta \omega}, \\
\delta \omega
\end{pmatrix}
_{\omega_g} = -\lambda
\frac{(\omega^{-1}\psi^2, \delta \omega)_{\omega_g}}{(\psi, \psi)_{\omega_g}}
\]

- Derivative of eigenvalues and eigenfunctions of \(Qf_n = \lambda_n \bar{U}(\omega)f_n\) w.r.t. \(\omega\)

\[
Q \frac{\partial f_n}{\partial \omega_j} = \frac{\partial \lambda_n}{\partial \omega_j} \bar{U} f_n + \lambda_n \frac{\partial \bar{U}}{\partial \omega_j} f_n + \lambda_n \bar{U} \frac{\partial f_n}{\partial \omega_j}
\]

\[
f_n^T \bar{U} f_n = 1
\]

- Since \(f_n^T \bar{U} f_n = 1\) and \(f_n^T (Q - \lambda_n \bar{U}) = 0\) [Nelson’76]

\[
\frac{\partial \lambda_n}{\partial \omega_j} = -\lambda_n f_n^T \frac{\partial \bar{U}}{\partial \omega_j} f_n \quad \text{and} \quad (Q - \lambda_n \bar{U}) \frac{\partial f_n}{\partial \omega_j} = \frac{\partial \lambda_n}{\partial \omega_j} \bar{U} f_n + \lambda_n \frac{\partial \bar{U}}{\partial \omega_j} f_n
Surface Mapping in the Embedding Space

Discretization

- The energy discretization

\[
E(\omega_1, \omega_2) = \sum_{n=1}^{N} \left( \frac{1}{S_1} \left( \frac{f_{1,n}}{\sqrt{\lambda_{1,n}}} - \frac{f_{2,n}(u_1)}{\sqrt{\lambda_{2,n}}} \right)^T U_1 \left( \frac{f_{1,n}}{\sqrt{\lambda_{1,n}}} - \frac{f_{2,n}(u_1)}{\sqrt{\lambda_{2,n}}} \right) + \frac{1}{S_2} \left( \frac{f_{2,n}}{\sqrt{\lambda_{2,n}}} - \frac{f_{1,n}(u_2)}{\sqrt{\lambda_{1,n}}} \right)^T U_2 \left( \frac{f_{2,n}}{\sqrt{\lambda_{2,n}}} - \frac{f_{1,n}(u_2)}{\sqrt{\lambda_{1,n}}} \right) \right)
\]

- Update weights iteratively

\[
\frac{\partial E}{\partial \omega_1} = 2 \sum_{n=1}^{N} \left[ \frac{1}{S_1} \left( \frac{1}{\sqrt{\lambda_{1,n}}} \frac{\partial f_{1,n}}{\partial \omega_1} - \frac{\partial \lambda_{1,n}}{\partial \omega_1} \frac{(f_{1,n})^T}{2 \sqrt[3]{\lambda_{1,n}}} \right) U_1 \left( \frac{f_{1,n}}{\sqrt{\lambda_{1,n}}} - \frac{Af_{2,n}}{\sqrt{\lambda_{2,n}}} \right) - \frac{1}{S_2} \left( \frac{\partial f_{1,n}}{\partial \omega_1} \frac{B^T}{\sqrt{\lambda_{1,n}}} - \frac{\partial \lambda_{1,n}}{\partial \omega_1} \frac{(Bf_{1,n})^T}{2 \sqrt[3]{\lambda_{1,n}}} \right) U_2 \left( \frac{f_{2,n}}{\sqrt{\lambda_{2,n}}} - \frac{Bf_{1,n}}{\sqrt{\lambda_{1,n}}} \right) \right] + 2\xi Q_1 \omega_1
\]

\[
\frac{\partial E}{\partial \omega_2} = 2 \sum_{n=1}^{N} \left[ \frac{1}{S_2} \left( \frac{1}{\sqrt{\lambda_{2,n}}} \frac{\partial f_{2,n}}{\partial \omega_2} - \frac{\partial \lambda_{2,n}}{\partial \omega_2} \frac{(f_{2,n})^T}{2 \sqrt[3]{\lambda_{2,n}}} \right) U_2 \left( \frac{f_{2,n}}{\sqrt{\lambda_{2,n}}} - \frac{Bf_{1,n}}{\sqrt{\lambda_{1,n}}} \right) - \frac{1}{S_1} \left( \frac{\partial f_{2,n}}{\partial \omega_2} \frac{A^T}{\sqrt{\lambda_{2,n}}} - \frac{\partial \lambda_{2,n}}{\partial \omega_2} \frac{(Af_{2,n})^T}{2 \sqrt[3]{\lambda_{2,n}}} \right) U_1 \left( \frac{f_{1,n}}{\sqrt{\lambda_{1,n}}} - \frac{Af_{2,n}}{\sqrt{\lambda_{2,n}}} \right) \right] + 2\xi Q_2 \omega_2
\]
Hippocampal Mapping Results

- Two hippocampal surfaces
  - Use 30 eigenfunctions in constructing the embedding space
  - Start with constant weights
- The weight of the source mesh are updated iteratively to compensate for the non isometric differences

4, 5, 6, 7-th eigenfunctions after conf. deform

Before: Large distortion

After: high quality map

Resulting conf. deform.
Cortical mapping example

Source Surface $M_1$  
Optimized weight $w$

Target Surface $M_2$  
Energy

Angle Distortion

Metric Distortion
Optimized Embedding

Original EF on $M_1$

Original EF on $M_2$

Optimized EF on $M_1$

$3^{rd}$ EF

$6^{th}$ EF
Group-wise atlas construction

• Given a set of annotated surfaces $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_p$, estimate a group-wise atlas $(\mathcal{M}^*, \omega^* g)$ for cortical label fusion

• Variational formula

$$\arg \min_\omega \sum_{i=1}^{p} \int_{\mathcal{M}^*} \left( d_{\Phi_i}^* (x, \mathcal{M}_i) \right)^2 d\mathcal{M}^* + \sum_{i=1}^{p} \int_{\mathcal{M}_p} \left( d_{\Phi_i}^* (\mathcal{M}^*, x) \right)^2 d\mathcal{M}_p$$
Clinical Applications of Hippocampal Mapping

- Population study: hippocampal atrophy in multiple sclerosis (MS) patients with depression
  - 109 female patient split into two groups with the CES-D scale: low depression (CES-D≤20) and high depression (CES-D>20)
  - Statistically significant group differences were localized on hippocampus (P=0.019)
  - Correlates well with clinical measure of depression
Folding free global conformal mapping [Lai-Wen-Yin-Gu-Lui]

- The variational formula for harmonic maps of genus-0 surfaces.

\[
\min_{\vec{F}=(f_1,f_2,f_3)} \mathcal{E}(\vec{F}) = \frac{1}{2} \int_\mathcal{M} \sum_{i=1}^{3} \| \nabla_{\mathcal{M}} f_i \|_2^2 d\mathcal{M}, \quad s.t. \quad \| \vec{F}(x) \|_2^2 = 1, \quad \forall x \in \mathcal{M}.
\]

- Slow convergence of gradient projection method.
- Possible foldings for surfaces with long and sharp features.

Our new algorithms.
- Speed up the computation efficiency tremendously using curvilinear search method [Wen-Glodfarb-Yin] or SOC [Lai-Osher].
- Can obtain folding-free maps using LB eigensystem for conformal deformed surfaces.
### Algorithm 2: Folding removal by weighted LB Eigen-projection

1. Compute a map \( F_0 : \mathcal{M} \rightarrow S^2 \) using Algorithm 1. Compute the corresponding \( e^{2u_0} \) using the approximation formula (14). Iterate the following steps starting from \( k = 1 \).
2. Given the map \( F_{k-1} \) and conformal factor \( e^{2u_{k-1}} \), solve (20).
3. Construct a Star map using \( \{\phi_1^{u_{k-1}}, \phi_2^{u_{k-1}}, \phi_3^{u_{k-1}}\} \).
4. Start Algorithm 1 for the Star map and obtain \( F_k \) and \( u_k \).
Folding free global conformal mapping [Lai-Wen-Yin-Lui-Gu]

Table 3: Comparison between the proposed Algorithm 2 and the algorithm in [1].
Computing the conformal and topological spectra of Riemannian Surfaces  
(Ongoing project with Chiu-Yen Kao and Braxton Osting)

- The Conformal spectrum.
  Let \([g_0]\) be the conformal class of \(g_0\) with fixed \(\text{vol}(\mathcal{M}, g) = 1\). The k-th conformal LB eigenvalue of \((\mathcal{M}, g_0)\) is:

\[
\lambda_k^C(\mathcal{M}, [g_0]) = \sup_{g \in [g_0]} \lambda_k(\mathcal{M}, g) = \sup \{ \lambda_k(\mathcal{M}, g) \text{vol}(\mathcal{M}, g) \mid g \text{ conformal to } g_0 \}
\]

- The topological spectrum of a 2-dim closed surface with genus \(\gamma\).

\[
\lambda_k^T(\gamma) = \sup_{g \in G(\mathcal{M}_\gamma)} \lambda_k(\mathcal{M}_\gamma, g).
\]

- In our work:

  1. For a given Riemannian manifold \((\mathcal{M}, g_0)\), we develop a computational method for finding a metric \(g \in [g_0]\) which attains the conformal spectrum.

  2. We also develop a computational method for studying the topological spectrum for genus \(\gamma = 0\) and \(\gamma = 1\), which depends on a parameterization of moduli space.

  3. We study the flat tori and embedded tori.
A new variational method for computing conformal factor [Kao-Lai-Osting]

- All genus-0 surfaces are conformal equivalent.
- The conformal factor for a genus-0 surface \( M \) satisfies the following highly nonlinear equation (Yamabe problem):

\[
\Delta_M u + \tilde{K} e^{2u} - K = 0
\]

whose solution is not straightforward to obtain. (\( \tilde{K} = 1 \) for the unit sphere)

- The optimizer of \( \lambda^C_1(M, [g_0]) = \sup_{g \in [g_0]} \lambda_1(M, g) \) is the canonical metric \( \mathbb{S}^2 \) [Hersch’70]

Conformal factor of Homer Simpson
Conclusion and future work

We consider the surface conformal deformation and study the Laplace-Beltrami eigenproblem on surfaces with conformal deformation

- Conformal deformation for registration between non isometric surfaces. We demonstrate its applications on surfaces from Brain image. It can be used to more general registration problems in computer graphics.

- Folding free global conformal maps.

- Computing conformal and topological spectrum

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