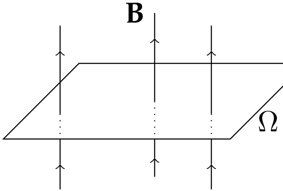
Sharp Estimates on the Magnetic and Pauli Spectra of Plane Domains

Richard Laugesen
University of Illinois at Urbana–Champaign

(joint with Bartłomiej Siudeja, *University of Oregon*)



SIAM Annual Meeting
San Diego, July 8, 2013

Lower bound for first Laplace eigenvalue

$-\Delta u = \lambda u$ on plane domain, Dirichlet boundary condition $u = 0$

Eigenvalues

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty$$

How does the *shape* of the domain affect its eigenvalues?

Rayleigh–Faber–Krahn (1920s)

$$\lambda_1(\text{domain}) \ge \lambda_1(\text{disk of same area})$$

Scale-invariant form $\boxed{\lambda_1 A \text{ is minimal for disk}}$

Methods: symmetric decreasing rearrangement, and more…

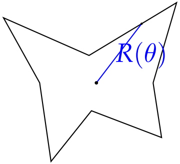
Upper bound for first Laplace eigenvalue

Assume domain Ω is **starlike** (*e.g.* convex)

Pólya–Szegő (1951; higher dim. by Freitas–Krejčiřík 2008)

$$\boxed{\lambda_1 A/G_0 \text{ is maximal for disk}}$$

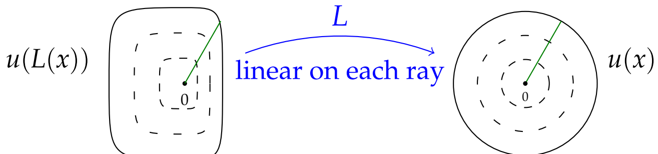
where $\boxed{G_0 \geq 1}$ is a scale-invariant geometric factor that penalizes deviation of Ω from a disk



$$G_0 = \frac{1}{2\pi}\int_0^{2\pi} \left[1 + (\log R)'(\theta)^2\right] d\theta$$

$$\geq 1 \quad \text{with equality for disk}$$

Method: transplant radial eigenfn $u(x)$ of disk to trial fn on Ω



No angular information in L and so it cannot handle higher eigenvalues.

LOWER BOUND FOR FIRST MAGNETIC EIGENVALUE

Magnetic field $(0, 0, \beta/A) = \nabla \times (-x_2, x_1, 0)\frac{\beta}{2A}$, flux$= \beta$,
vector potential is $F = \frac{\beta}{2A}(-x_2, x_1)$. Complex-valued wavefun u.

Magnetic Laplacian:

$\boxed{(i\nabla + F)^2 u = Eu}$ on plane domain, boundary condition $u = 0$

Eigenvalues (energies of charged quantum particle)

$$0 < E_1 \leq E_2 \leq E_3 \leq \cdots \to \infty$$

How does the *shape* affect the eigenvalues?

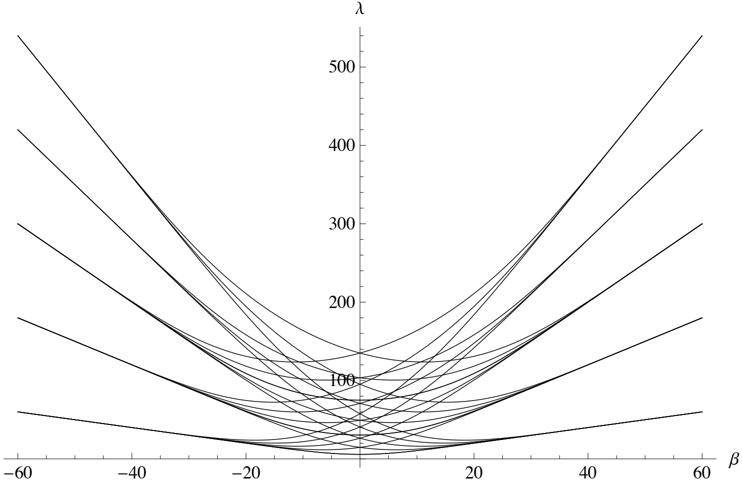
L. Erdös (1996)

$$E_1(\text{domain}) \geq E_1(\text{disk of same area})$$

Scale-invariant form $\boxed{E_1 A \text{ is minimal for disk}}$

Method: clever rearrangements…

MAGNETIC SPECTRUM OF THE UNIT DISK



The double eigenvalues of the disk "split" when the magnetic field is turned on. Also notice the "clumping" of energies for large magnetic field (showing emergence of Landau levels).

UPPER BOUND FOR FIRST MAGNETIC EIGENVALUE

$$(i\nabla + F)^2 u = Eu, \quad \text{boundary condition } u = 0$$

Define

$$G_0 = \frac{1}{2\pi}\int_0^{2\pi} [1 + (\log R)'(\theta)^2]\, d\theta \geq 1, \qquad G_1 = \frac{2\pi I}{A^2} \geq 1,$$

where $I = \int_\Omega |x|^2\, dA =$ moment of inertia about origin. Then
G_0 measures oscillation of boundary,
G_1 measures elongation of boundary. Let

$$\boxed{G = \max\{G_0, G_1\} \geq 1}$$

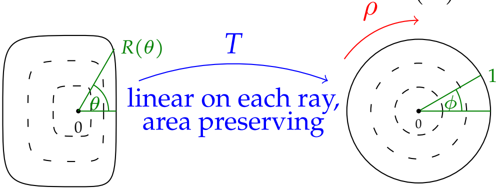
with $G = 1$ for disk.

Theorem (Laugesen & Siudeja, in preparation)

Among starlike plane domains, the normalized magnetic ground state energy $E_1 A/G$ is ***maximized*** *when the domain is a centered disk.*

Proof that $E_1 A/G$ is maximal for disk: Assume $A(\Omega) = \pi$.

trial function v
$= u(r/R(\theta), \phi(\theta) - \rho)$



eigenfn. u

Area-preserving means $\boxed{R(\theta)^2\, d\theta = d\phi}$, or $\phi'(\theta) = R(\theta)^2$.

Use

$$E_1(\Omega) \le R[v] \equiv \frac{\int_\Omega |(i\nabla + F)v|^2\, dx}{\int_\Omega |v|^2\, dx}.$$

Need to evaluate this Rayleigh quotient by changing variable back to the disk.

Trial function $v(r,\theta) = u(r/R(\theta), \phi(\theta) - \rho)$ has Rayleigh quotient

$$R[v] = \int_\Omega |(i\nabla + F)v|^2 \, dA = Q_1 + Q_2 + Q_3$$

where

$$\mathbf{Q_1} = \int_0^{2\pi} \int_0^1 \left| u_s(s, \phi(\theta) - \rho) \right|^2 s ds \left[1 + (\log R)'(\theta)^2 \right] d\theta$$

$$\mathbf{Q_2} = 2\mathrm{Re} \int_0^{2\pi} \int_0^1 \overline{u_s(s, \phi(\theta) - \rho)} \quad \times$$
$$\Big(-\frac{1}{s} u_\phi(s, \phi(\theta) - \rho) + \frac{i\beta}{2\pi} s u(s, \phi(\theta) - \rho) \Big) s ds \, R(\theta) R'(\theta) \, d\theta$$

$$\mathbf{Q_3} = \int_0^{2\pi} \int_0^1 \left| i \frac{1}{s} u_\phi(s, \phi(\theta) - \rho) + \frac{\beta}{2\pi} s u(s, \phi(\theta) - \rho) \right|^2 s ds \, R(\theta)^4 \, d\theta$$

(Use polar coordinates, chain rule, radial change of variable, and $\phi' = R^2$.) Now average w.r.t. all rotations $\rho \in [0, 2\pi]$ of the eigenfn.

Integrate over rotations $\rho \in [0, 2\pi]$:

$$\frac{1}{2\pi}\int_0^{2\pi} Q_1 \, d\rho = G_0(\Omega) \int_{\mathbb{D}} |u_s|^2 \, dx$$

$$\frac{1}{2\pi}\int_0^{2\pi} Q_2 \, d\rho = 0$$

$$\frac{1}{2\pi}\int_0^{2\pi} Q_3 \, d\rho = G_1(\Omega) \int_{\mathbb{D}} \left| i\frac{1}{s}u_\phi + \frac{\beta}{2\pi} s u \right|^2 dx$$

where $x = (x_1, x_2)$ has polar coordinates s, ϕ.

(Integrate, Fubinate, change $\rho \mapsto \phi(\theta) - \phi$, and separate the ρ and θ integrals.
For Q_2, notice that $\int_0^{2\pi} R(\theta)R'(\theta)\, d\theta = 0$ by periodicity.)

Since $G_0 \leq G$ and $G_1 \leq G$ by definition, we obtain

$$(\rho\text{-average of } Q_1 + Q_2 + Q_3) \leq G(\Omega)R[u] = G(\Omega)E_1(\mathbb{D})$$

$\square$

Eigenvalue functionals

Theorem (Laugesen & Siudeja, in preparation)

Among starlike plane domains, the following functionals are ***maximized*** *(for each $n \geq 1$) when the domain is a centered disk.*

- *fundamental tone:* $E_1 A/G$
- *sum of eigenvalues:* $(E_1 + \cdots + E_n)A/G$
- *sum of roots:* $(E_1^s + \cdots + E_n^s)^{1/s} A/G$ *for each* $0 < s \leq 1$
- *product of eigenvalues:* $\sqrt[n]{E_1 \cdots E_n} A/G$
- $\sum_{j=1}^n \Phi(E_j A/G)$*, for any concave increasing* Φ

Eigenvalue functionals

Theorem (Laugesen & Siudeja, in preparation)

Among starlike plane domains, the following functionals are ***maximized*** *(for each $n \geq 1$) when the domain is a centered disk.*

- *fundamental tone:* $E_1 A/G$
- *sum of eigenvalues:* $(E_1 + \cdots + E_n)A/G$
- *sum of roots:* $(E_1^s + \cdots + E_n^s)^{1/s} A/G$ *for each* $0 < s \leq 1$
- *product of eigenvalues:* $\sqrt[n]{E_1 \cdots E_n}\, A/G$
- $\sum_{j=1}^n \Phi(E_j A/G)$, *for any concave increasing* Φ

The following are ***minimized*** *when the domain is a centered disk*

- *partial sum of zeta function:* $\sum_{j=1}^n (E_j A/G)^s$ *for each* $s < 0$
- *partial sum of heat trace:* $\sum_{j=1}^n \exp(-E_j At/G)$ *for each* $t > 0$

[*Note.* Laplacian case $\beta = 0$ holds in *all* dimensions; see paper to appear in Journal of Spectral Theory, 2013.]

FROM SUMS TO HEAT TRACE BY MAJORIZATION (HARDY, LITTLEWOOD, PÓLYA)

If $a_1 \leq a_2 \leq a_3 \leq \cdots$ and $b_1 \leq b_2 \leq b_3 \leq \cdots$ and

$$a_1 + \cdots + a_n \leq b_1 + \cdots + b_n \qquad \forall n \geq 1$$

then

$$\Phi(a_1) + \cdots + \Phi(a_n) \leq \Phi(b_1) + \cdots + \Phi(b_n) \qquad \forall n \geq 1$$

for all concave increasing functions Φ.
(*Fun exercise.* Prove it for $n = 1, 2$.)

Example:
using $\Phi(c) = -\exp(-ct)$ shows heat trace is minimal for disk, in our theorem

EXTENSIONS

- Neumann boundary conditions? Yes, identical proof.
- Robin boundary conditions? Yes.
- Steklov eigenvalues?
  [Work in progress with A. Girouard]
- Quantum particle with spin:
  our theorems extend to the Pauli operator (under Dirichlet boundary condition), which is

  $$\big(\sigma \cdot (i\nabla + F)\big)^2 \psi = E\psi$$

  where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 3-tuple of Pauli matrices:

  $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  The Pauli operator acts on spinors, that is, on 2-component complex vector fields of the form $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$.

Open problems

- Simply connected domains, not necessarily starlike???
- Domains on sphere, or hyperbolic space???
- Higher dimensions: A is 1-form and $B = dA$ is 2-form. But the magnetic field breaks the symmetry, and so maximizer is presumably not the ball?
- Is Neumann Laplacian heat trace $\sum_{j=1}^{\infty} e^{-\mu_j A t}$ minimal for the disk, for each $t > 0$? True as $t \to 0, \infty$.
  (Luttinger proved disk is “maximal” for Dirichlet Laplacian.)

Open problems

- Simply connected domains, not necessarily starlike???
- Domains on sphere, or hyperbolic space???
- Higher dimensions: A is 1-form and $B = dA$ is 2-form. But the magnetic field breaks the symmetry, and so maximizer is presumably not the ball?
- Is Neumann Laplacian heat trace $\sum_{j=1}^{\infty} e^{-\mu_j A t}$ minimal for the disk, for each $t > 0$? True as $t \to 0, \infty$. (Luttinger proved disk is "maximal" for Dirichlet Laplacian.)

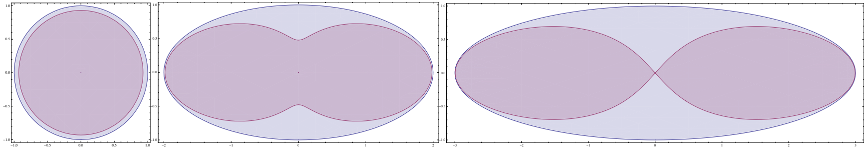
Conclusions

The method of area-preserving transformation and rotational averaging:

- is **geometrically sharp** — extremal domain is disk
- handles eigenvalue sums of arbitrary length (any n), and hence **spectral zeta functional** and **trace of heat kernel**
- **applies universally** — to Dirichlet, Robin and Neumann

Can both geometric factors play a role in $G = \max\{G_0, G_1\}$? Yes!

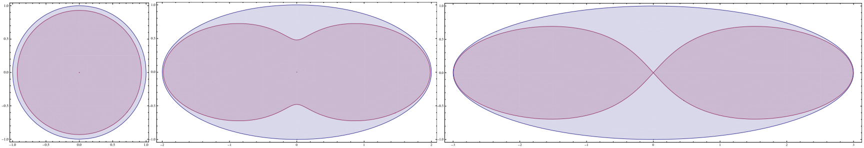
For an ellipse of large eccentricity, shifting the origin away from the center can result in either G_0 or G_1 dominating.



($G_0 < G_1$ when the origin lies in the shaded region)

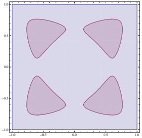
Can both geometric factors play a role in $G = \max\{G_0, G_1\}$? Yes!

For an ellipse of large eccentricity, shifting the origin away from the center can result in either G_0 or G_1 dominating.



($G_0 < G_1$ when the origin lies in the shaded region)

The square is different, with G_0 dominating for all origins near the center.



($G_0 < G_1$ when the origin lies in the shaded region)

Where is the best choice of origin?

The geometric factors depend on the choice of origin.

- To minimize G_1 ($\simeq$moment of inertia), we should choose the origin at the center of mass.

- To minimize $G_0 = \frac{1}{2\pi}\int_0^{2\pi}[1 + (\log R)'(\theta)^2]\,d\theta$, the best origin might *not* be the center of mass.

e.g. to minimize G_0 on a triangular domain we should choose the origin at the center of the inscribed circle, which can lie far from the center of mass.

Conclusion: No choice of origin will simultaneously minimize both of the geometric factors, in general.

Thus one should aim to choose the origin "somewhere near the center" in a way that minimizes the maximum of the two factors, $G = \max\{G_0, G_1\}$.