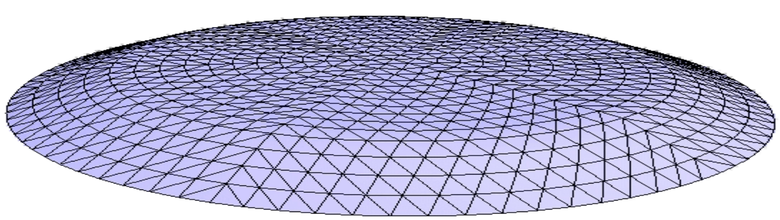
Spectral Shape Analysis with Applications in Medical Imaging



SIAM Annual Meeting 2013
San Diego

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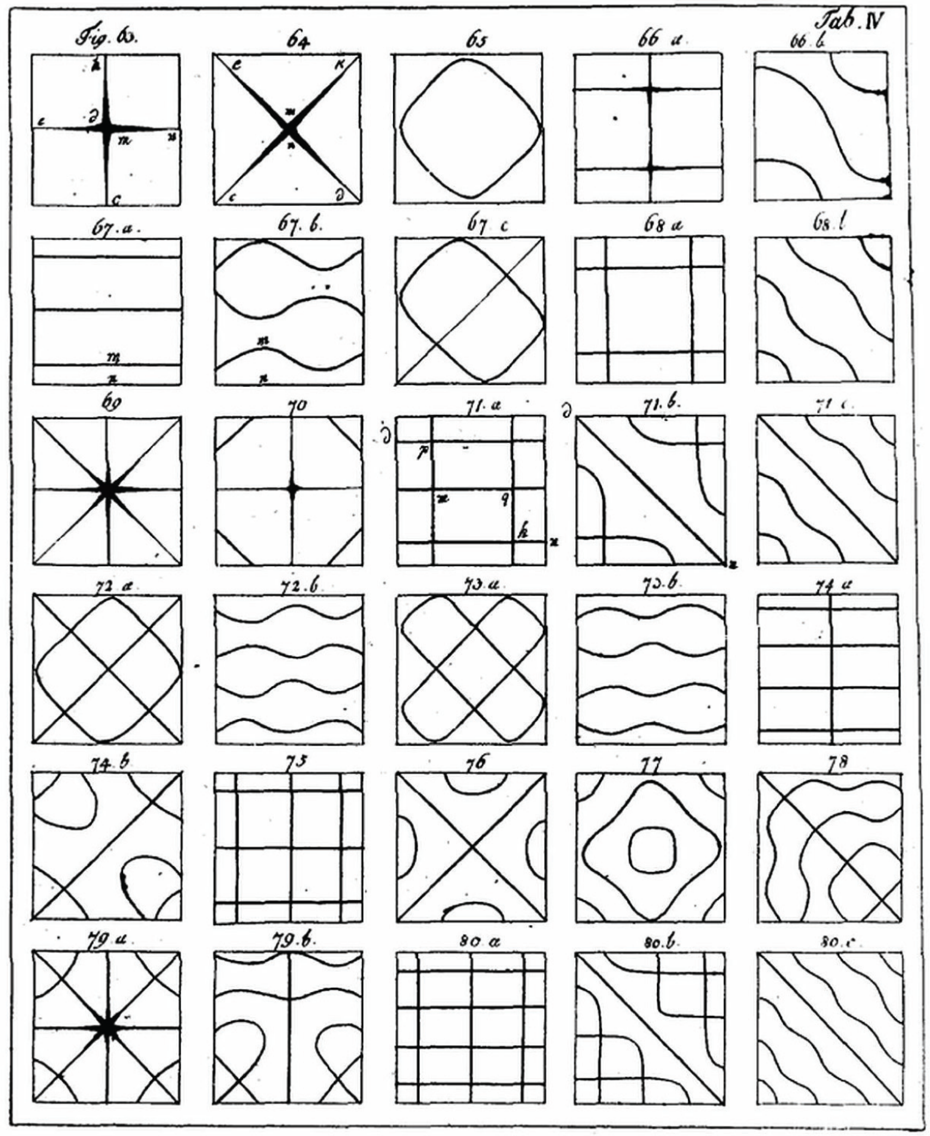
Sound and Shape

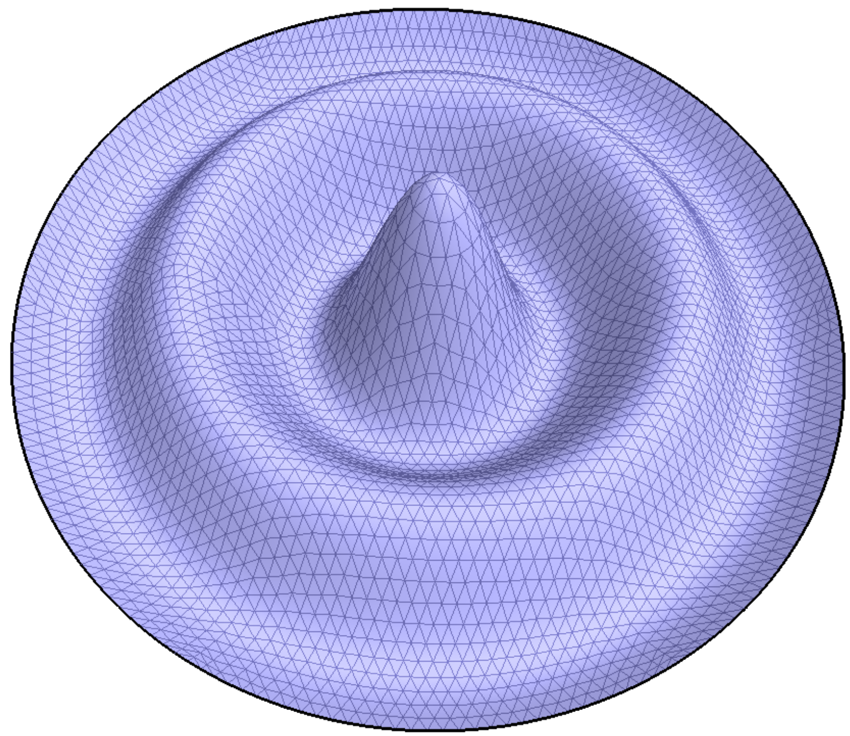
Chladni's strange patterns

- vibration of plates
- used a violin bow
- discovered sound patterns by spreading sand on the plates
- 1809 invited by Napoleon
- who held out price for mathematical explanation
- "Entdeckungen über die Theorie des Klanges" (Discoveries concerning the theory of music), Chladni, 1787



Can one "hear" Shape?

- First asked by Bers, then paper by Kac 1966, idea dates back to Weyl 1911 (at least).
- The sound (eigen-frequencies) of a drum depend on its shape.
- This spectrum can be numerically computed if the shape is known.
- E.g., no other shape has the same spectrum as a disk.
- Can the shape be computed from the spectrum?



Laplace Spectrum

Definition

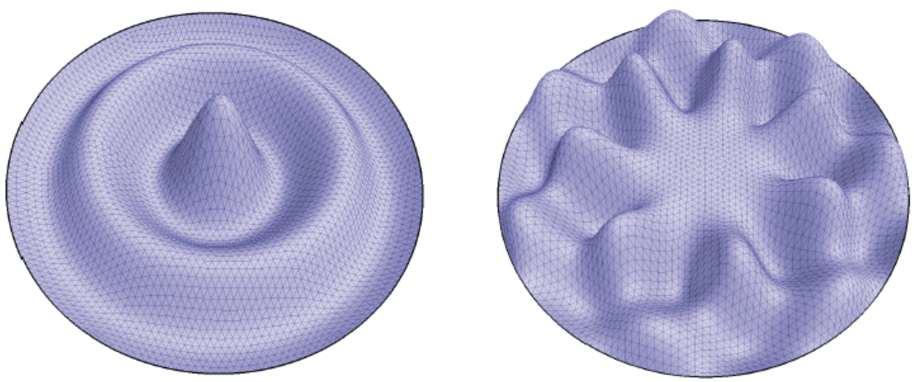
Helmholtz Equation (**Laplacian Eigenvalue Problem**):

$$\Delta f = -\lambda f, \qquad f : M \rightarrow \mathbb{R}$$

Solution: Eigenfunctions f_i with corresponding family of eigenvalues (**Spectrum**):

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty$$

Here Laplace-Beltrami Operator: $\Delta f := div(grad\ f)$

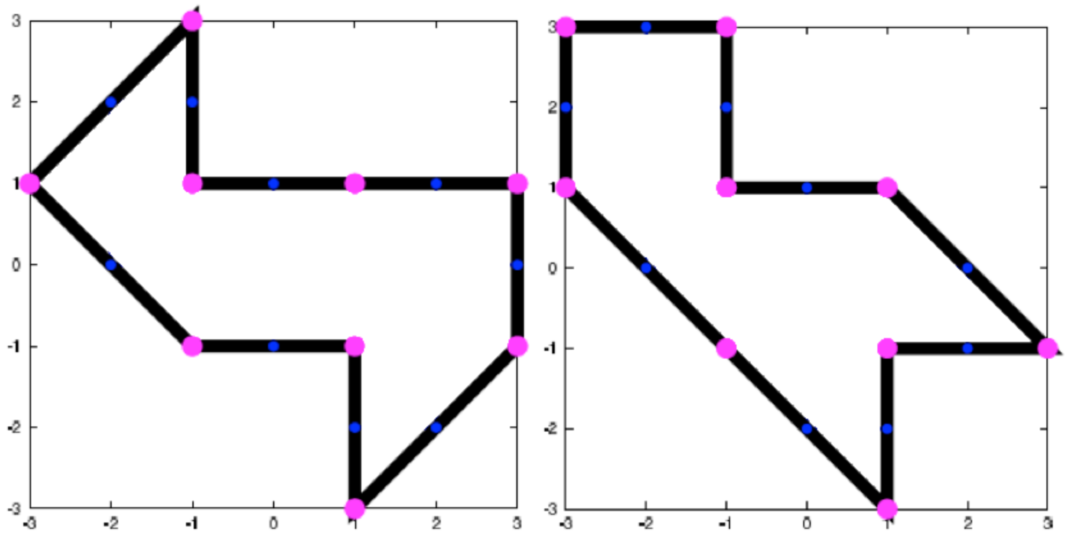


Can one hear Shape?

Answer

No! Isospectral drums exist (Gordon, Webb, Wolpert - 1992)

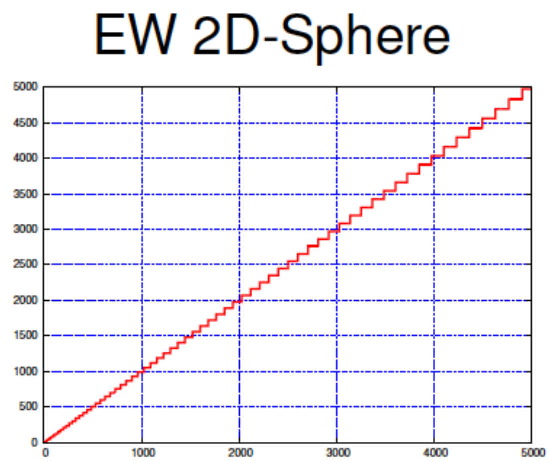
- rare
- concave in 2D
- only pairs

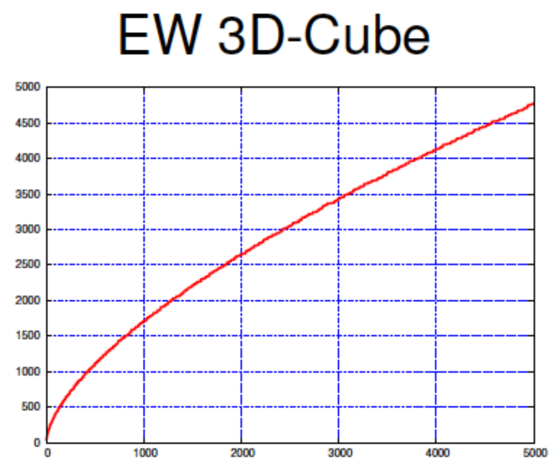


Geometry

Nevertheless, they share area, boundary length, genus...

Weyl's Theorem





Theorem (Weyl - 1911,1912)

$$\lambda_n \sim \frac{4\pi}{\mathrm{area}(D)}\, n \qquad \text{for } d = 2 \text{ and } n \to \infty$$

$$\lambda_n \sim \left(\frac{6\pi^2}{\mathrm{vol}(D)}\right)^{\frac{2}{3}} n^{\frac{2}{3}} \qquad \text{for } d = 3 \text{ and } n \to \infty.$$

Heat Trace Expension

- More Geometric and Toplogical Information:
- Riemannian Volume
- Riemannian Volume of the Boundary
- Euler Characteristic for closed 2D Manifolds
- Number of holes for planar domains
- Possible to extract data numerically from beginning sequence [reuter:06] (500 Eigenvalues)

Contributions

Shape Analysis: Reuter, Wolter, Peinecke [SPM05], [JCAD06](most cited paper award 09) and Patent Appl.

- Introduced Laplace-Beltrami Op. for Non-Rigid Shape Analysis.
- Cubic FEM to obtain accurate solutions for surfaces and solids.
- Before a mesh Laplace (simplified linear FEM) has been used for parametrization, smoothing, mesh compression

Image Recognition: Peinecke et al [JCAD07] (Mass Density LBO)

Neuroscience Applicantions:

- Statistical morphometric studies of brain structures (eigenvalues), Niethammer, Reuter, Shenton, … [MICCAI07], Reuter.. [CW08]
- Topological studies of eigenfunctions, Reuter.. [CAD09]

Segmentation: Reuter..(IMATI, Genova) [SMI09] [IJCV09]

Correspondence: Reuter [IJCV09]

Finite Element Method

- Instead of graph/wireframe (vertices, edges), we look at elements that assemble our geometry without gaps:
  - triangles
  - tetrahedra
  - voxels….
- We define basis functions over this discretized geometry (linear, quadratic, cubic …)
- We get a powerful framework to solve differential equations (not just Laplace).
- We also developed cubic FEM for parametrized surfaces (working in parameter space, similar to isoparametric FEM) [Reuter CAD06]

Higher Order

Theorem (Convergence)

For decreasing mesh size h and order p form functions: Eigenvalues converge with order

$$O(2p)$$

and Eigenfunctions with order

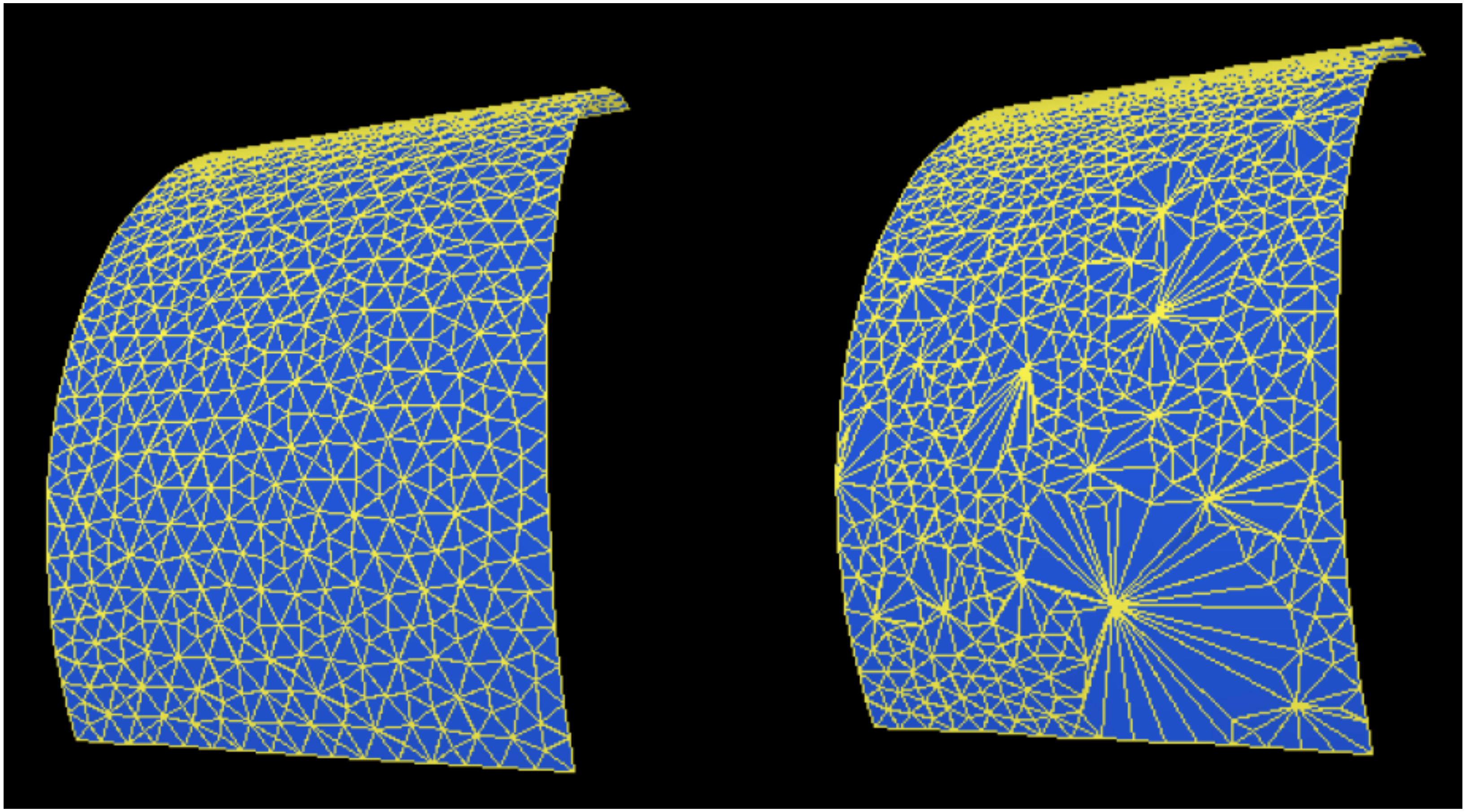
$$O(p+1)$$

in the L_2 norm .

$\Rightarrow$ Always prefer higher order FEM approximations over mesh refinement.

Uniform and Non-Uniform Mesh

[SMI09]



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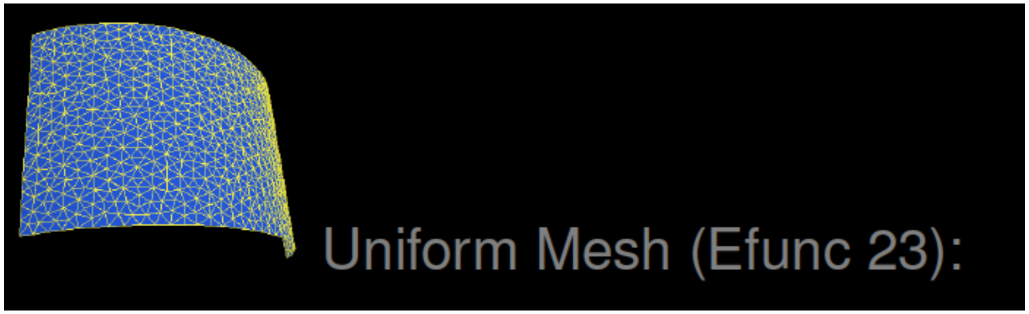
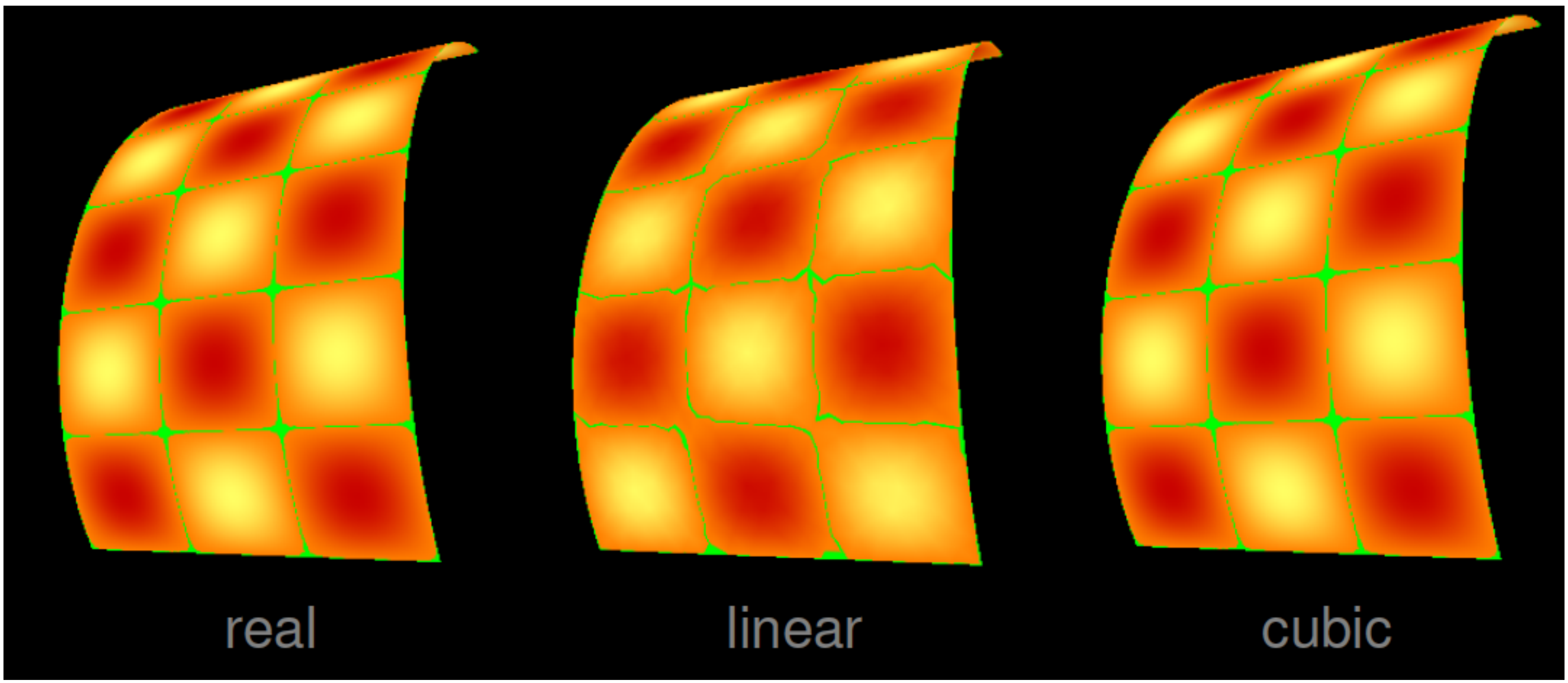
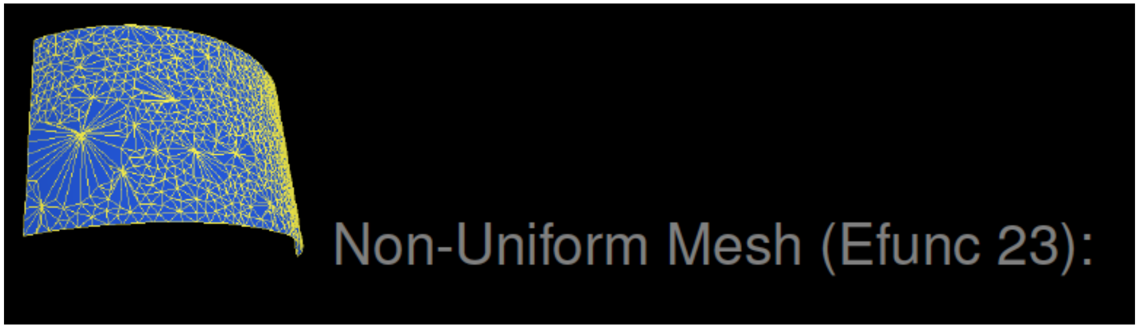
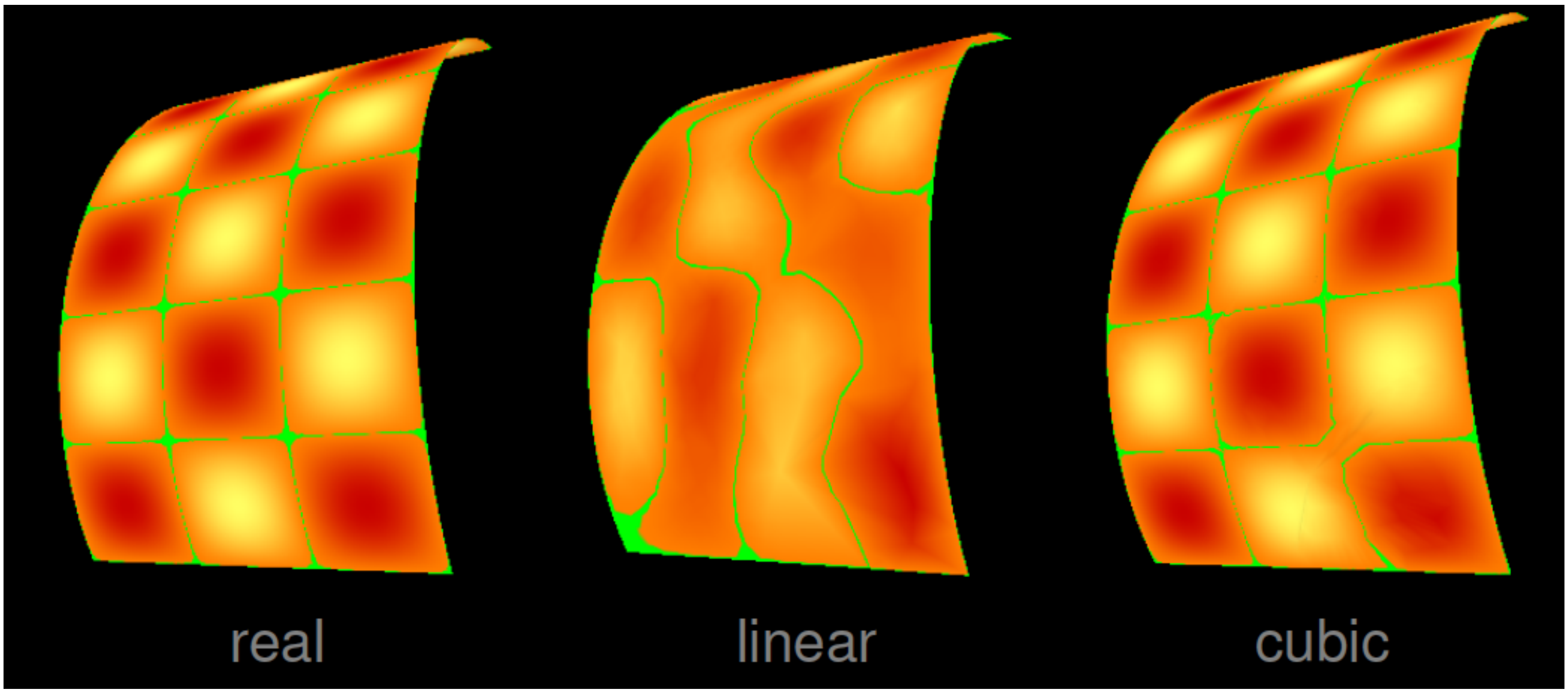




[SMI09]





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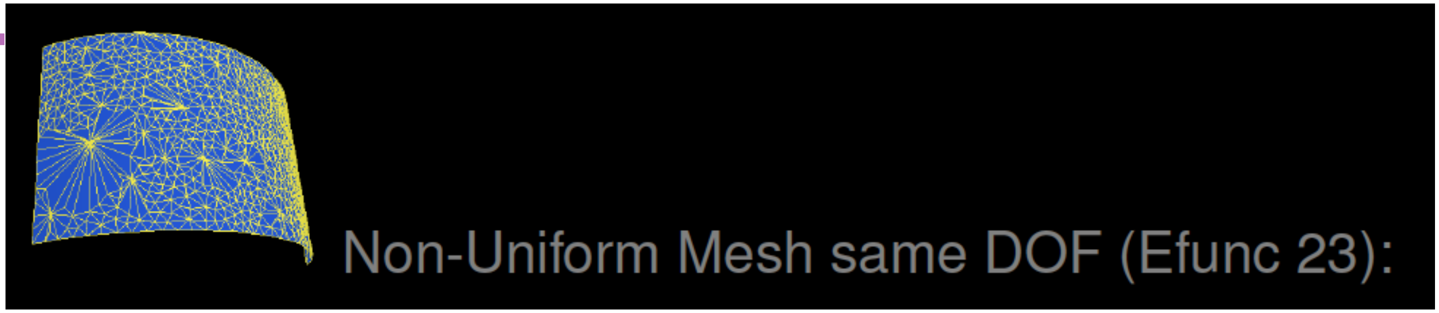
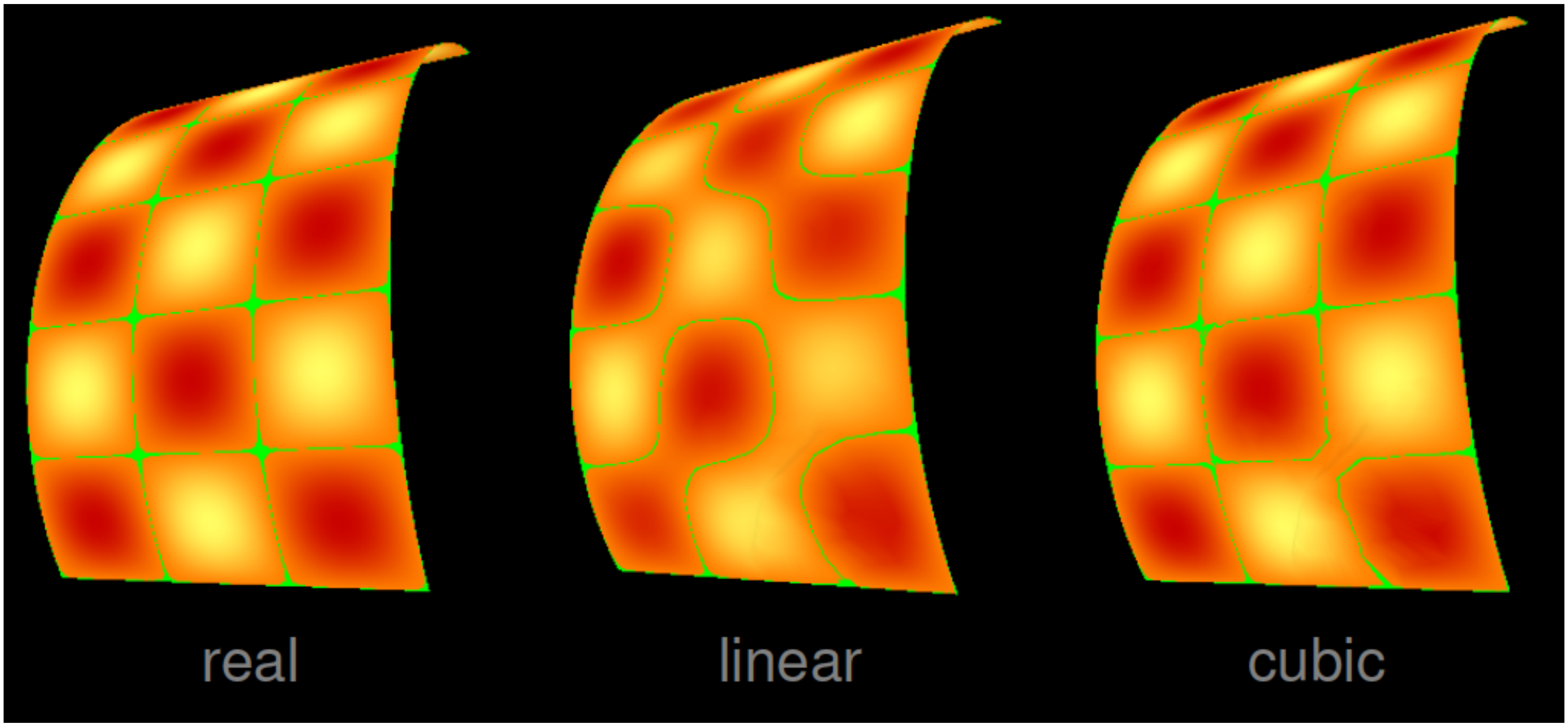
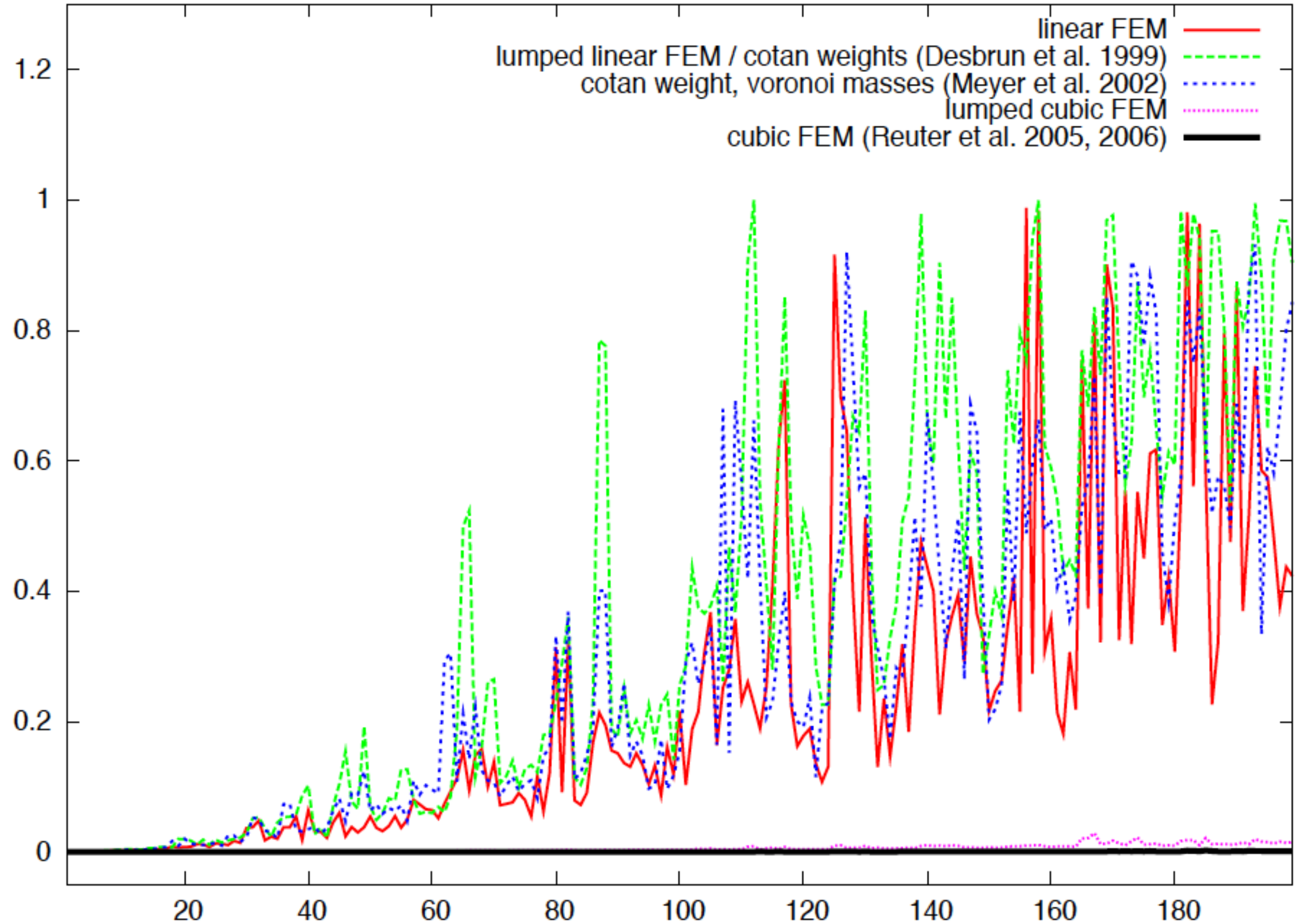




[SMI09]

Comparison Eigenfunctions

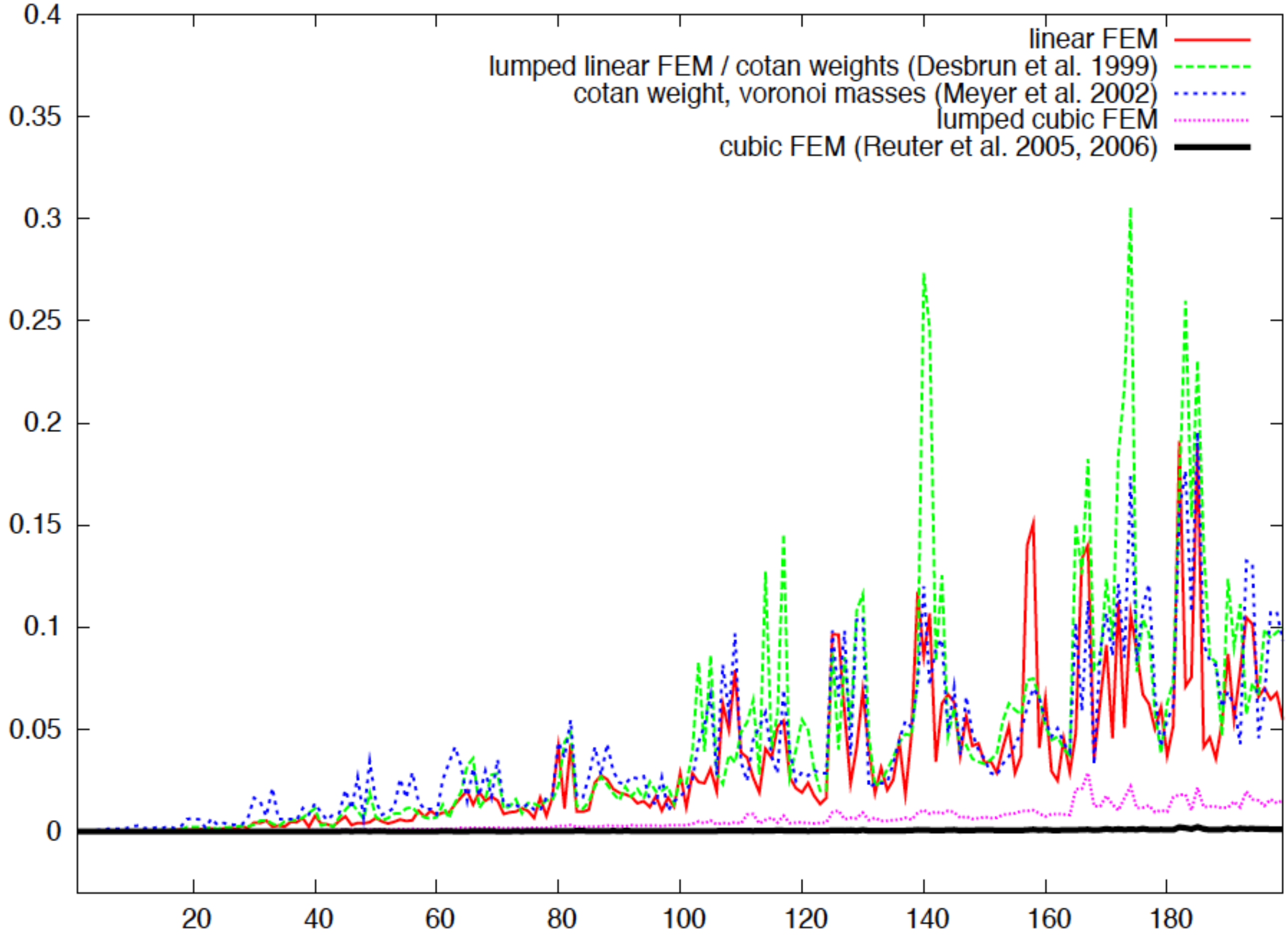
Rectangle - Uniform Mesh - first 200 Eigenfunctions:



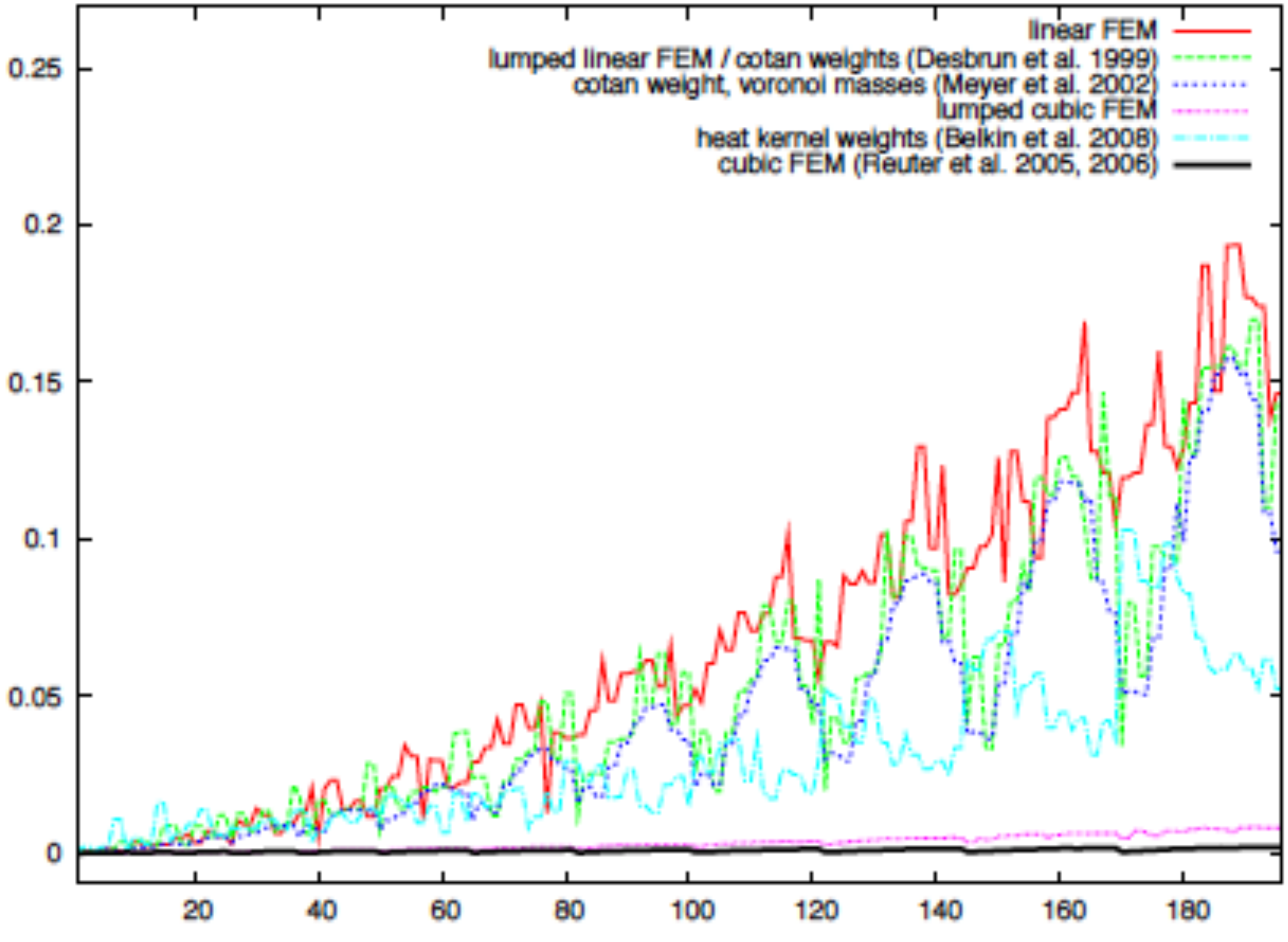
[SMI09]

Comparison Eigenfunctions

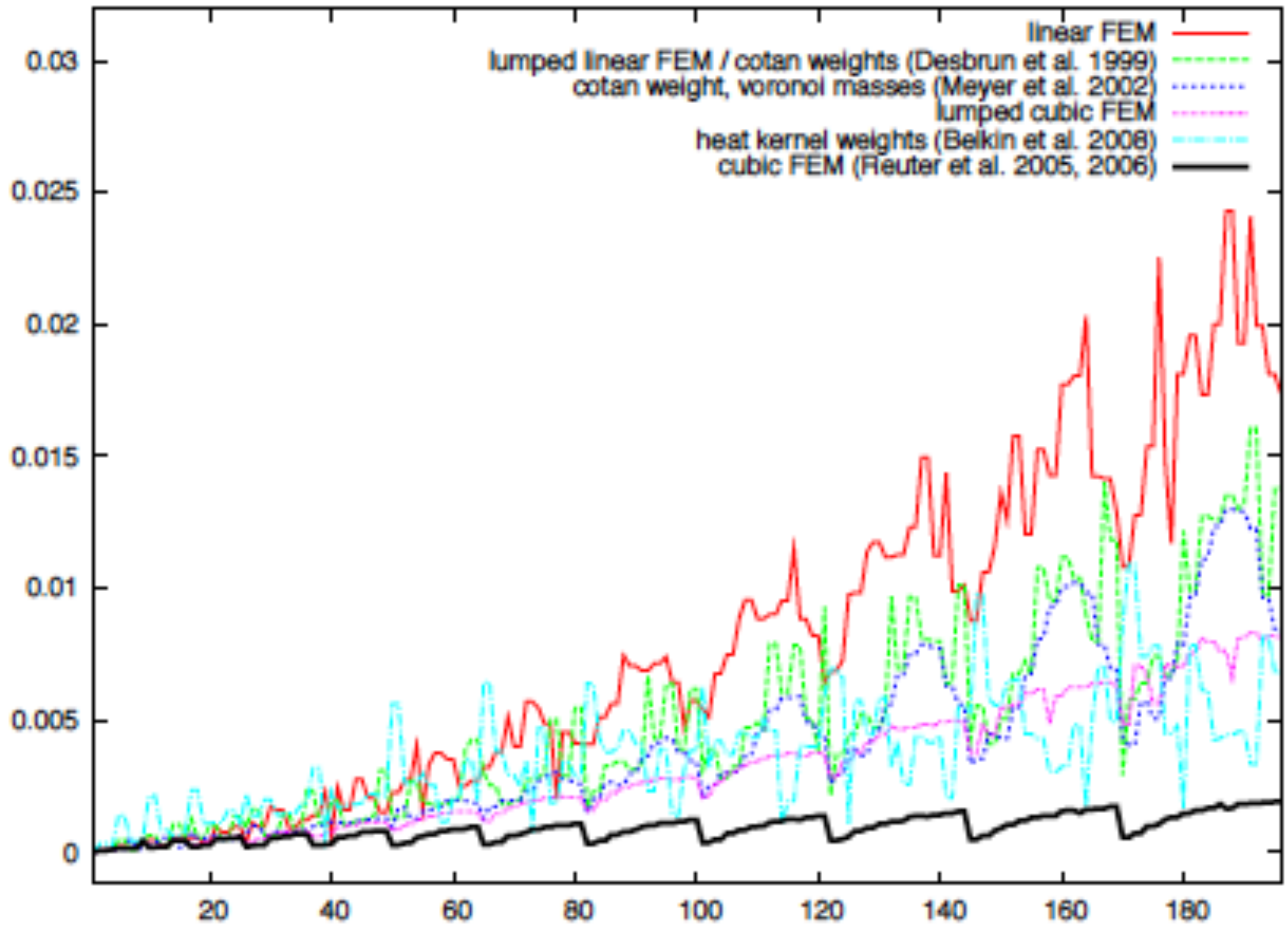
Rectangle - Uniform Mesh (same DOF as cubic) - 200 EF:



Comparison on the sphere

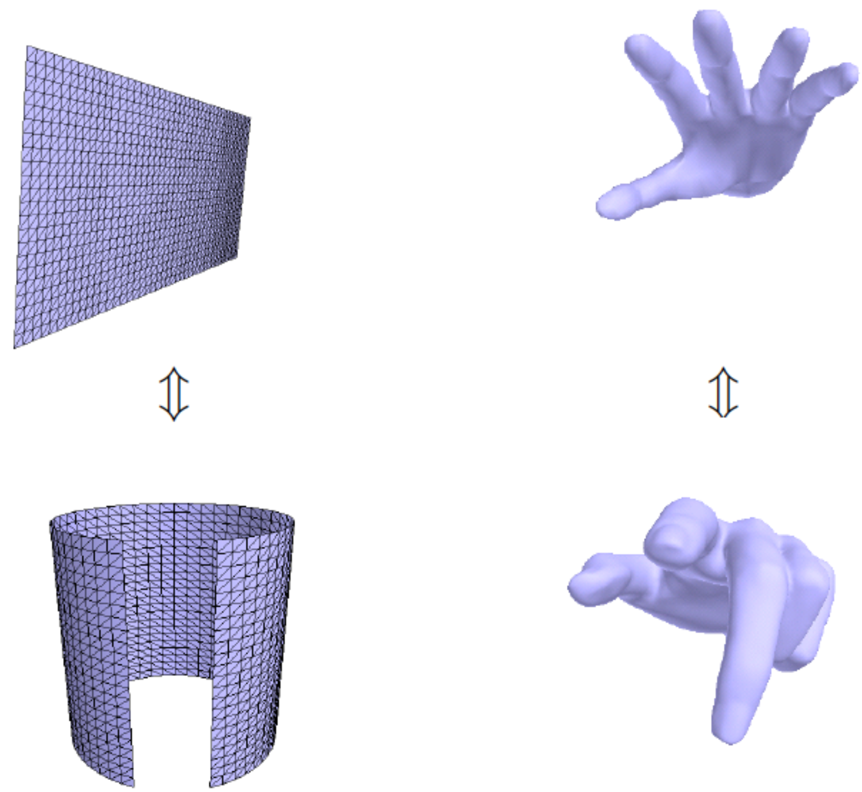


+ Sphere – same DOF

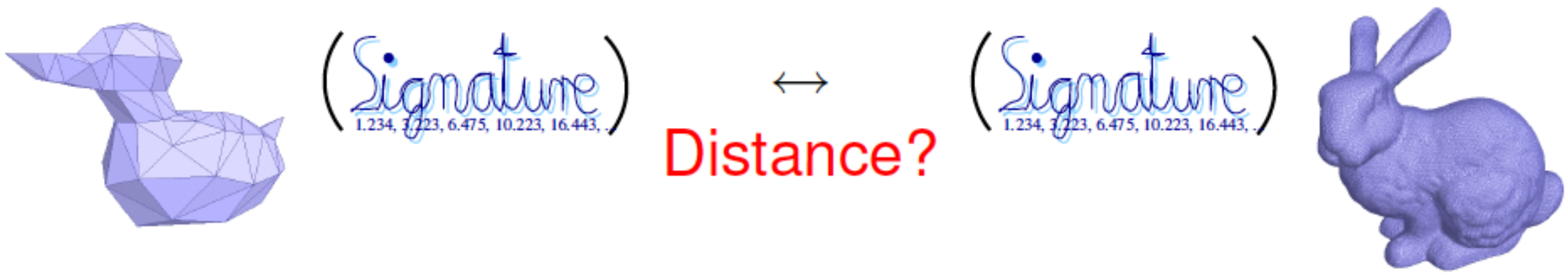


+ What is Shape and what is similar?

- Shape should be invariant with respect to:
  - Location (rotation, translation)
  - Size
  - Isometries?



Shape Matching



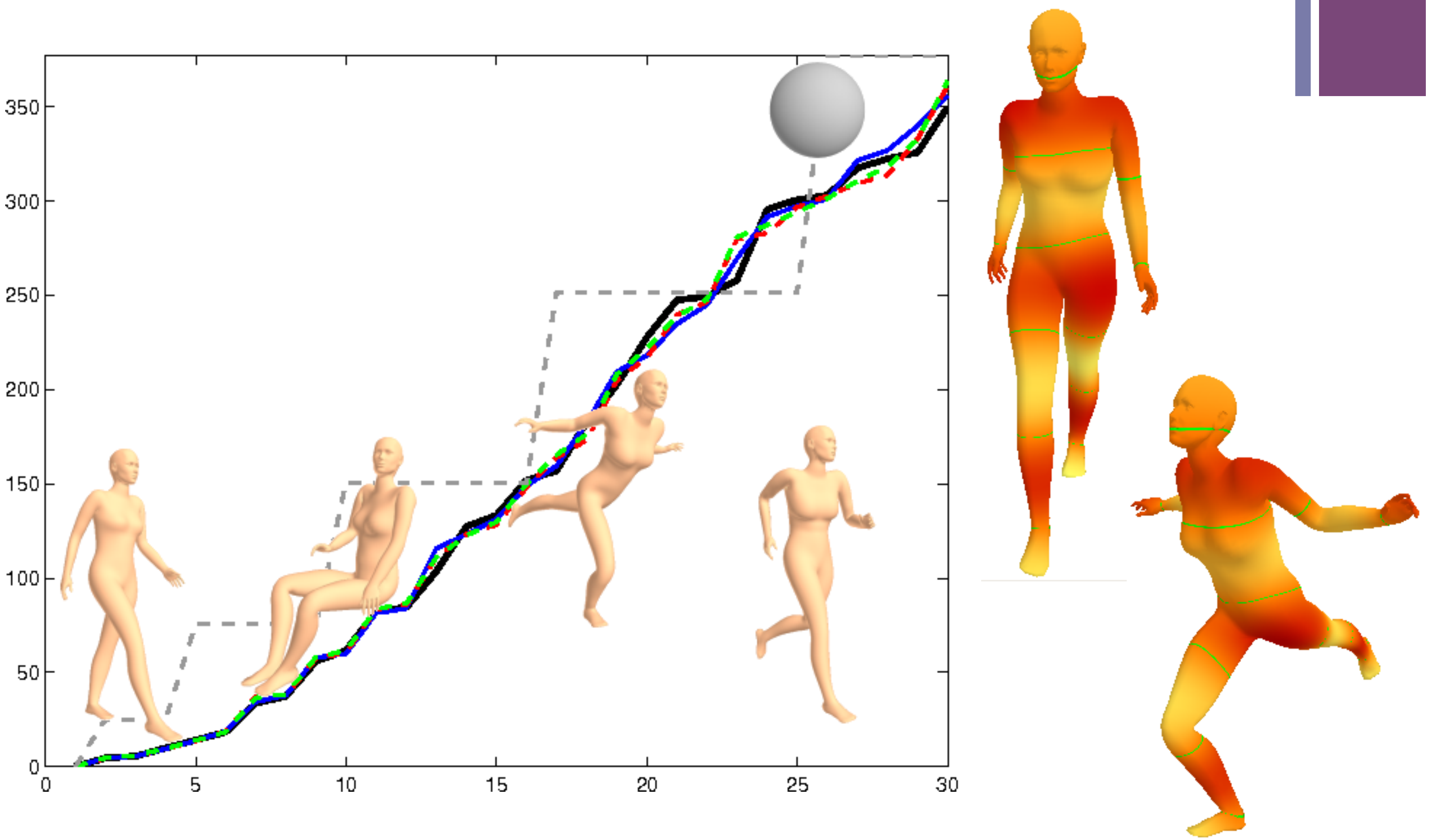
- Prior alignment, scaling of the objects:
  - normalization, registration
- Computation of a simplified representation
  - Signature, Shape-Descriptor
- Comparison of the signatures
  - distance computation to measure similarity
- Disadvantages of current methods:
  - Over-simplification, missing invariance, complex pre-processing, difficult to compare signatures, support only special representations

New Signature: ShapeDNA [spm05]

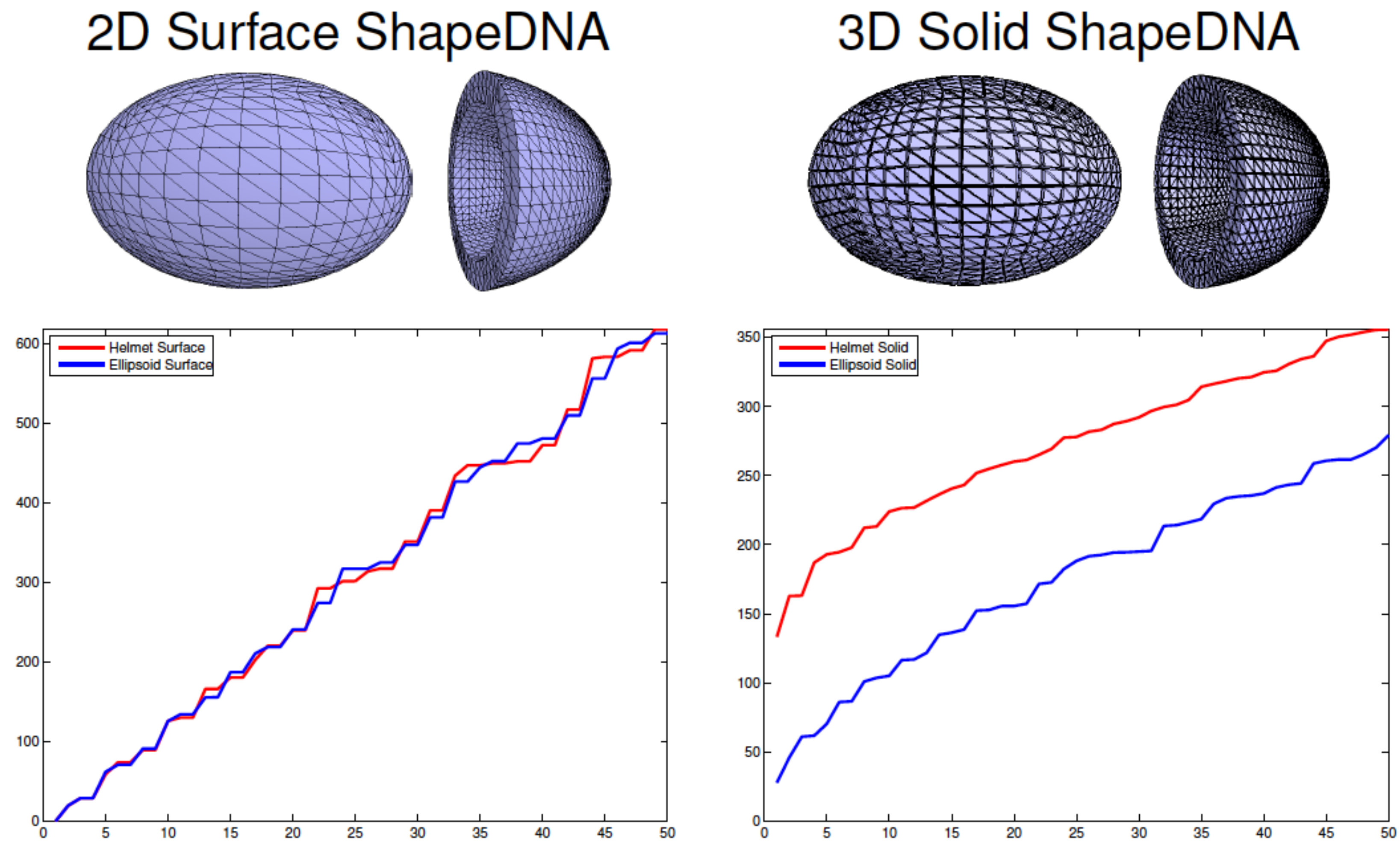
We use the (normed) n-dim vector of the **smallest** n **eigenvalues** $(\lambda_1, \ldots, \lambda_n)$ of the Laplace operator Δ as the signature:

- Isometry invariant $\Rightarrow$ location invariant
- and (where required) scaling invariant
- No registration necessary
- Surfaces & solids (even with cavities)
- Independent of representation
- Simple distance computation of the signature vectors
- Efficient and highly accurate computation with cubic FEM

Isometry Invariance

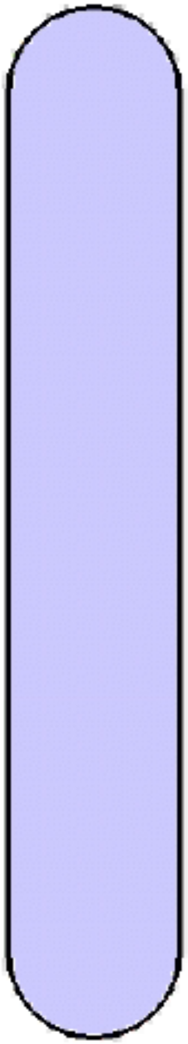


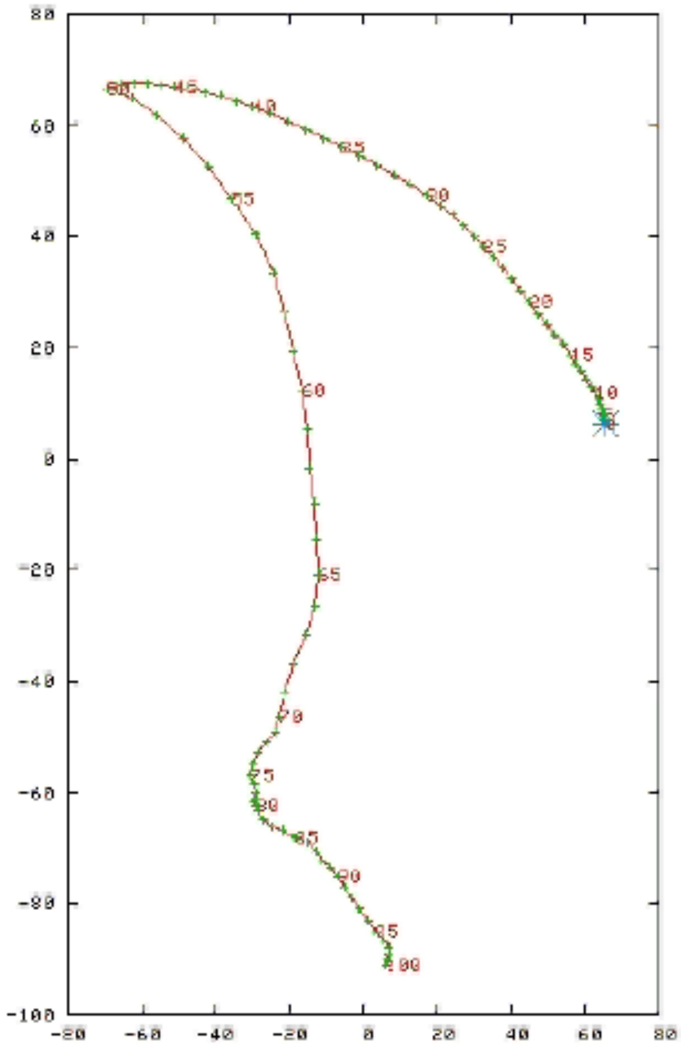
+ 2D near isometry, 3D not



For solid bodies in $\mathbb{R}^3$ isometry is equivalent to congruency.

Continuous Shape Dependence





M.Reuter

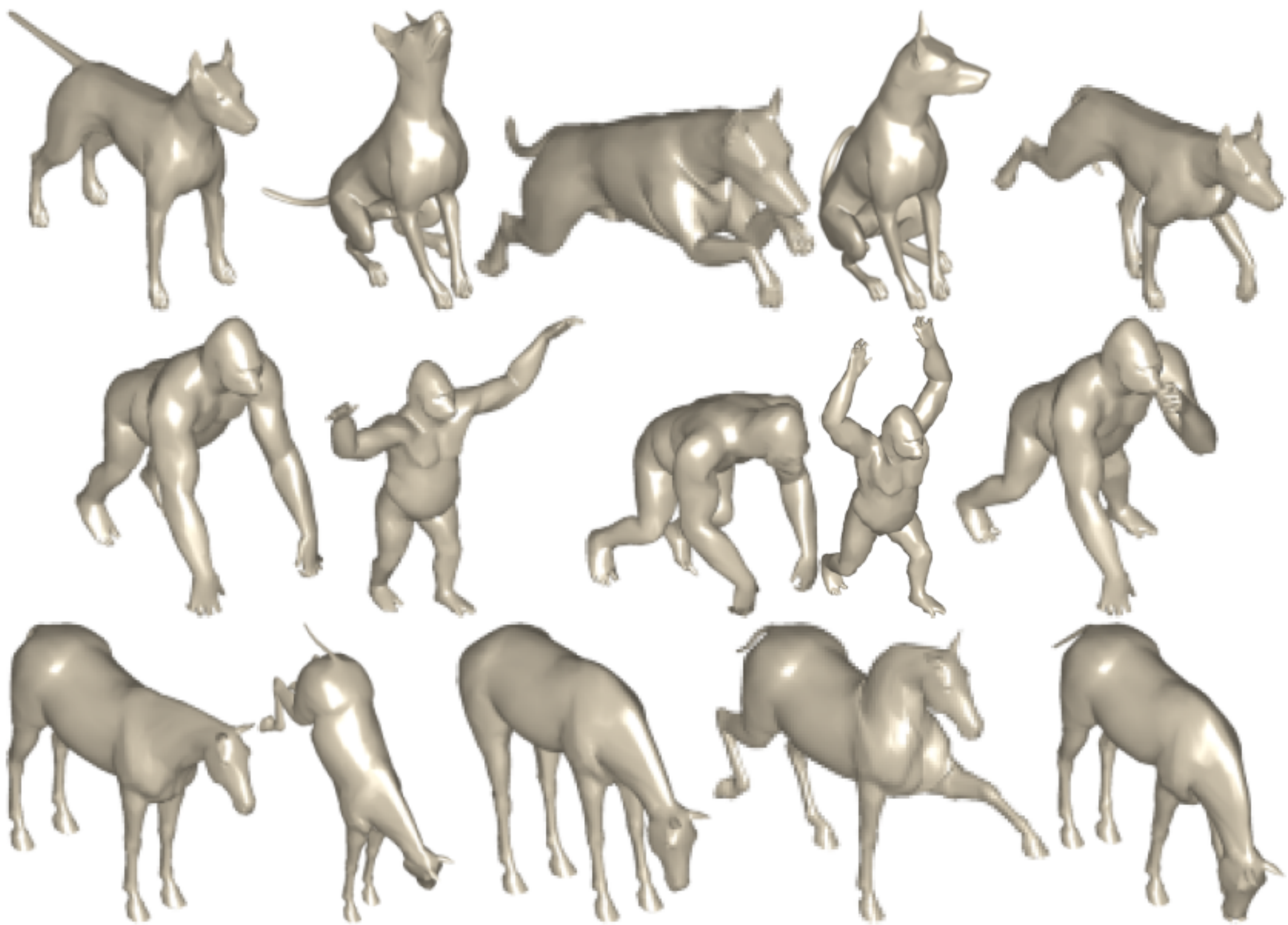
Database Retrieval

- 1. Computation of the first n Eigenvalues (Shape-DNA)
- 2. Normalization
  - a) Surface area normalized
  - b) Volume normalized
- 3. Distance computation of the Shape-DNA (n-dim vector)
  - a) Euclidean distance (←default)
  - b) Another p-norm
  - c) Hausdorff distance
  - d) Correlation . . .

Nonrigid Shape Database (148)

Courtesy of Bronstein, Bronstein, Kimmel, 2006

Nonrigid Shape Database (148)



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Nonrigid Shape Database (148)

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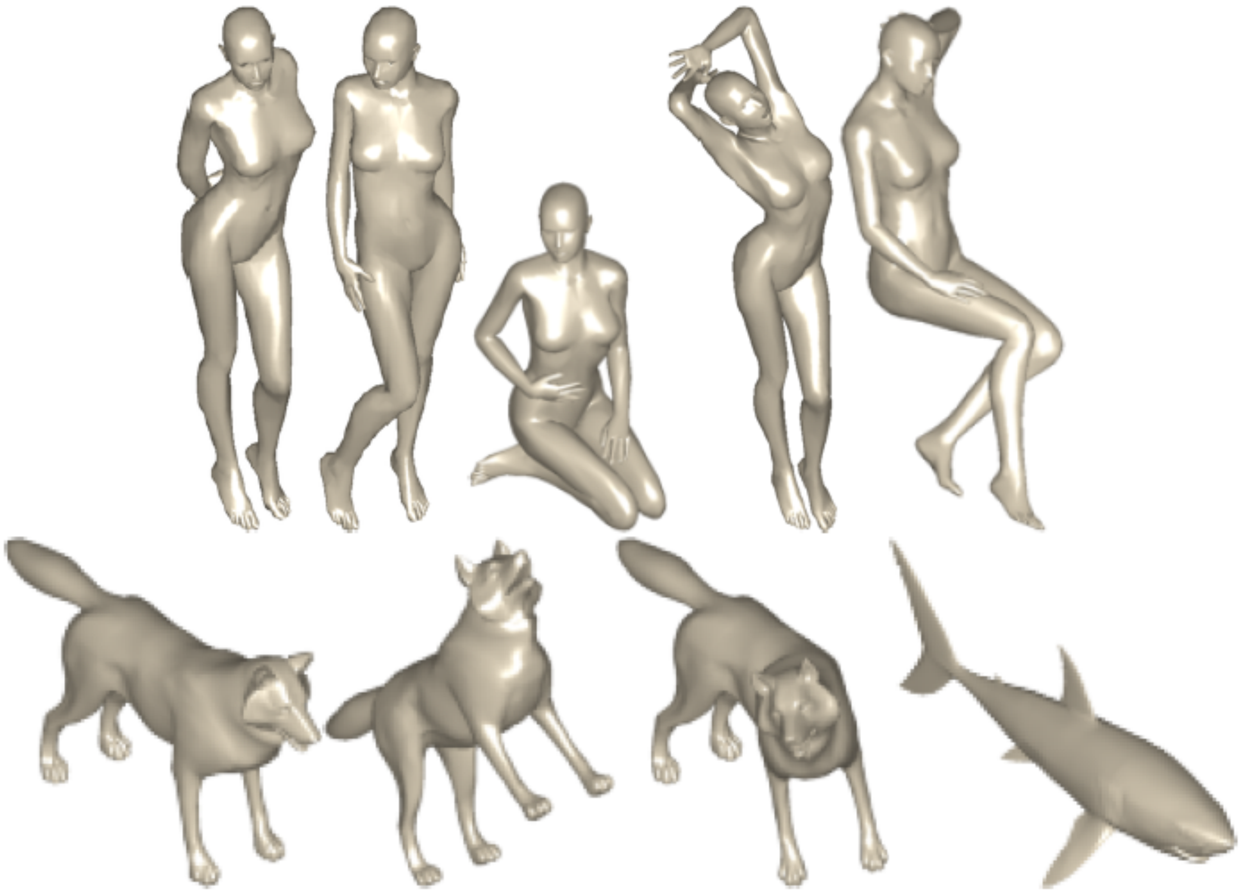
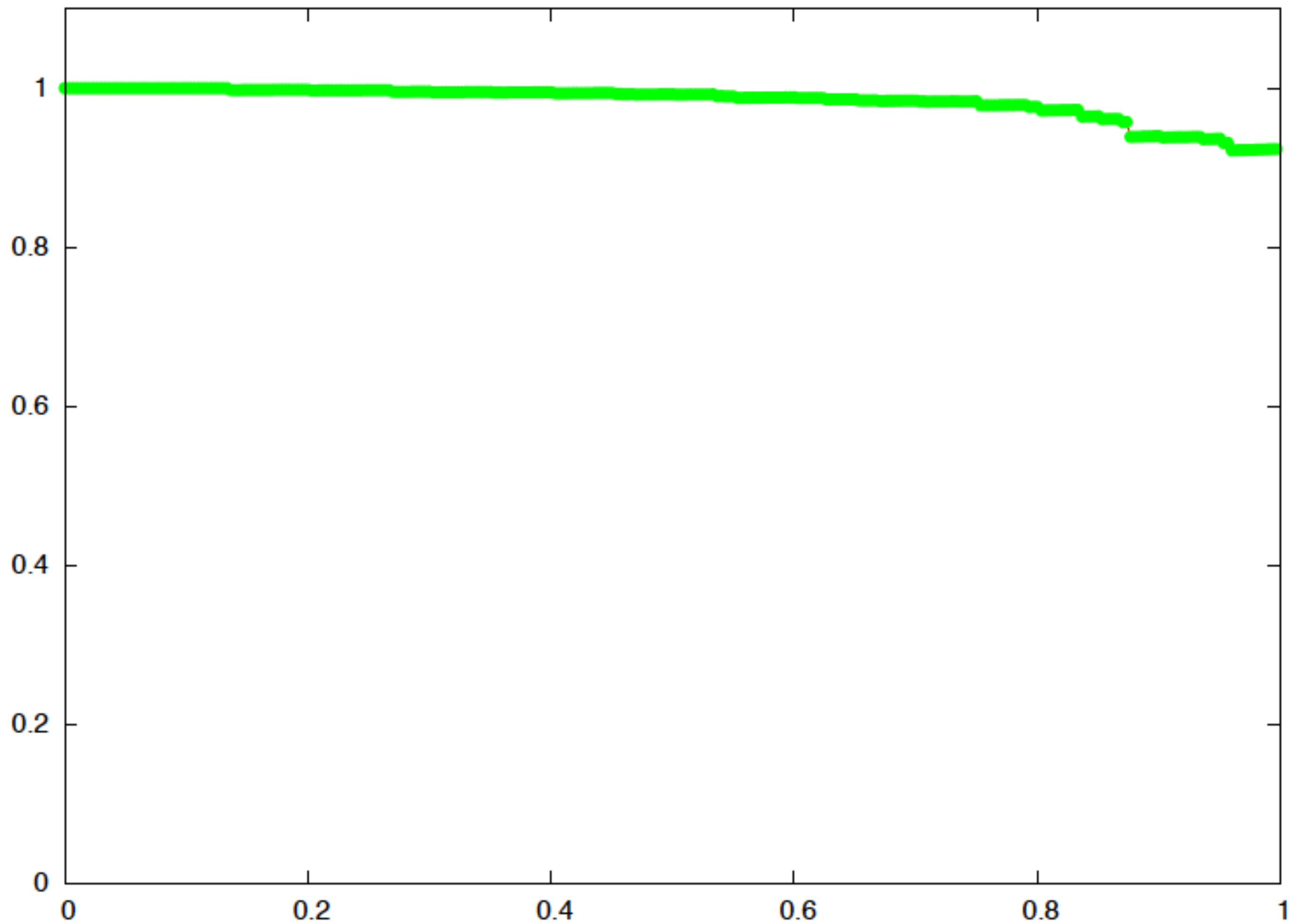


Courtesy of Bronstein, Bronstein, Kimmel, 2006

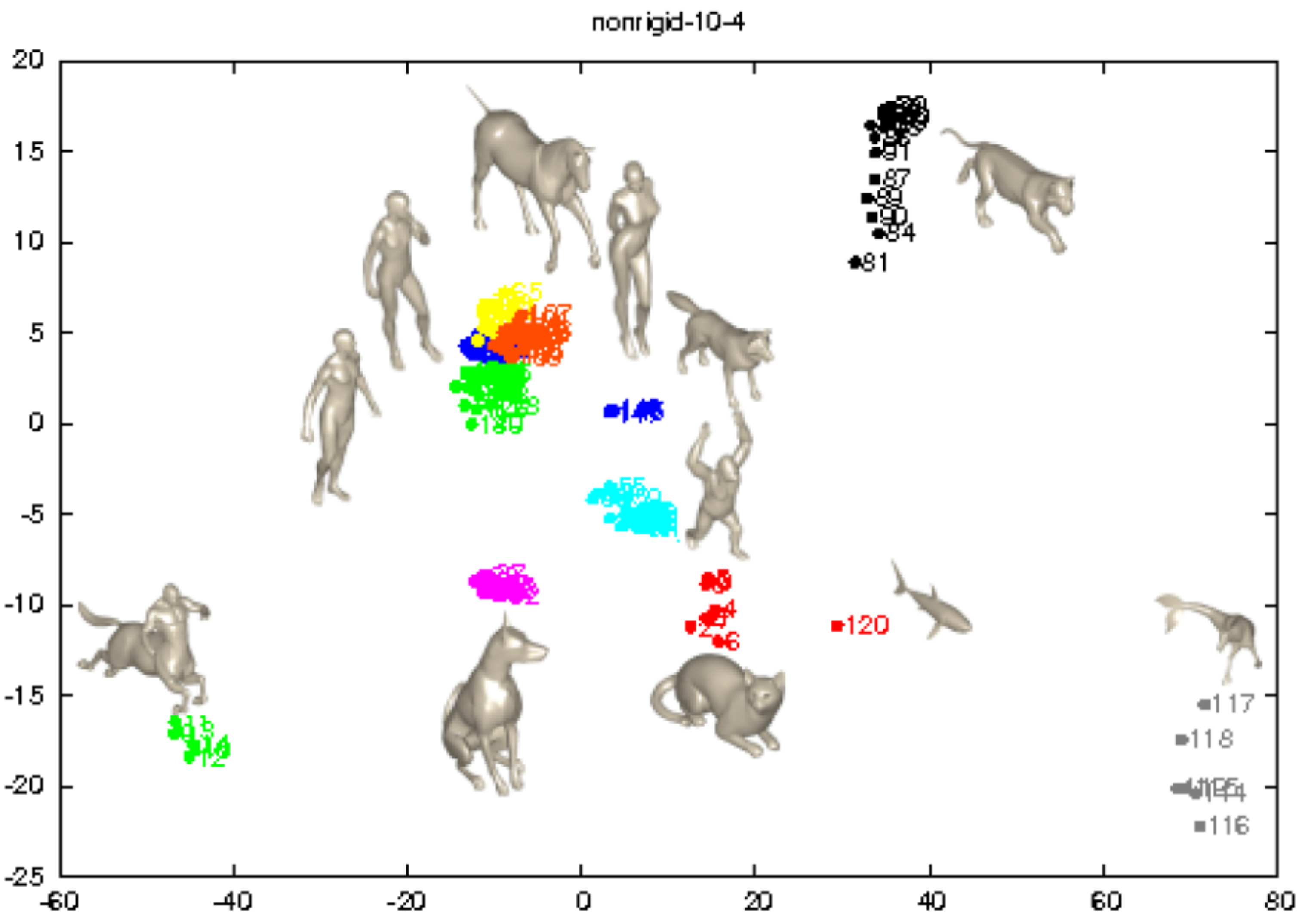
Nonrigid Shape Database (148)

$P(R) := Precision(Recall)$ averaged:

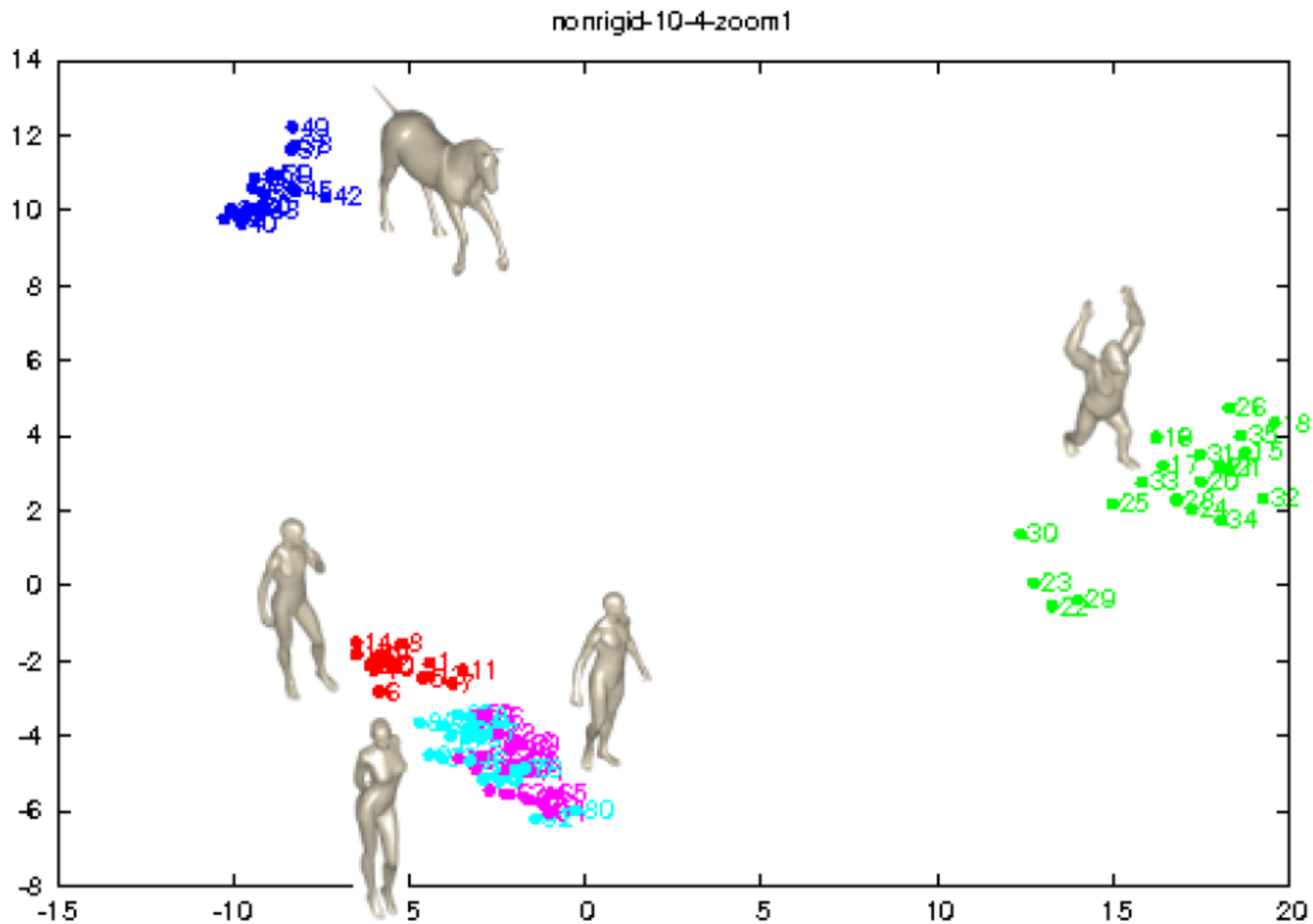


Integral: 0.983301 and min: 0.921847

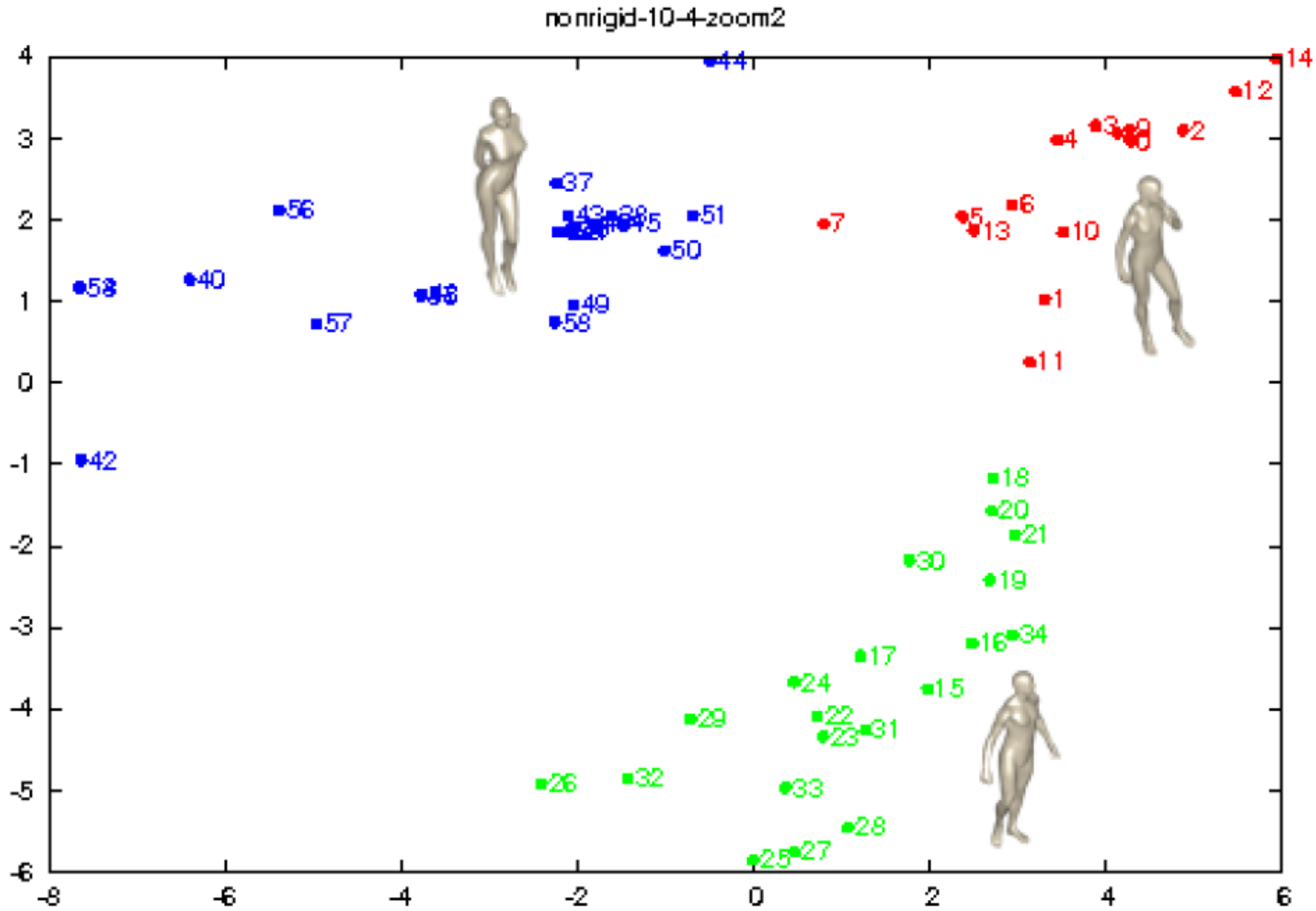
Nonrigid DB – MDS Plot



+ Nonrigid DB – Zoom 1

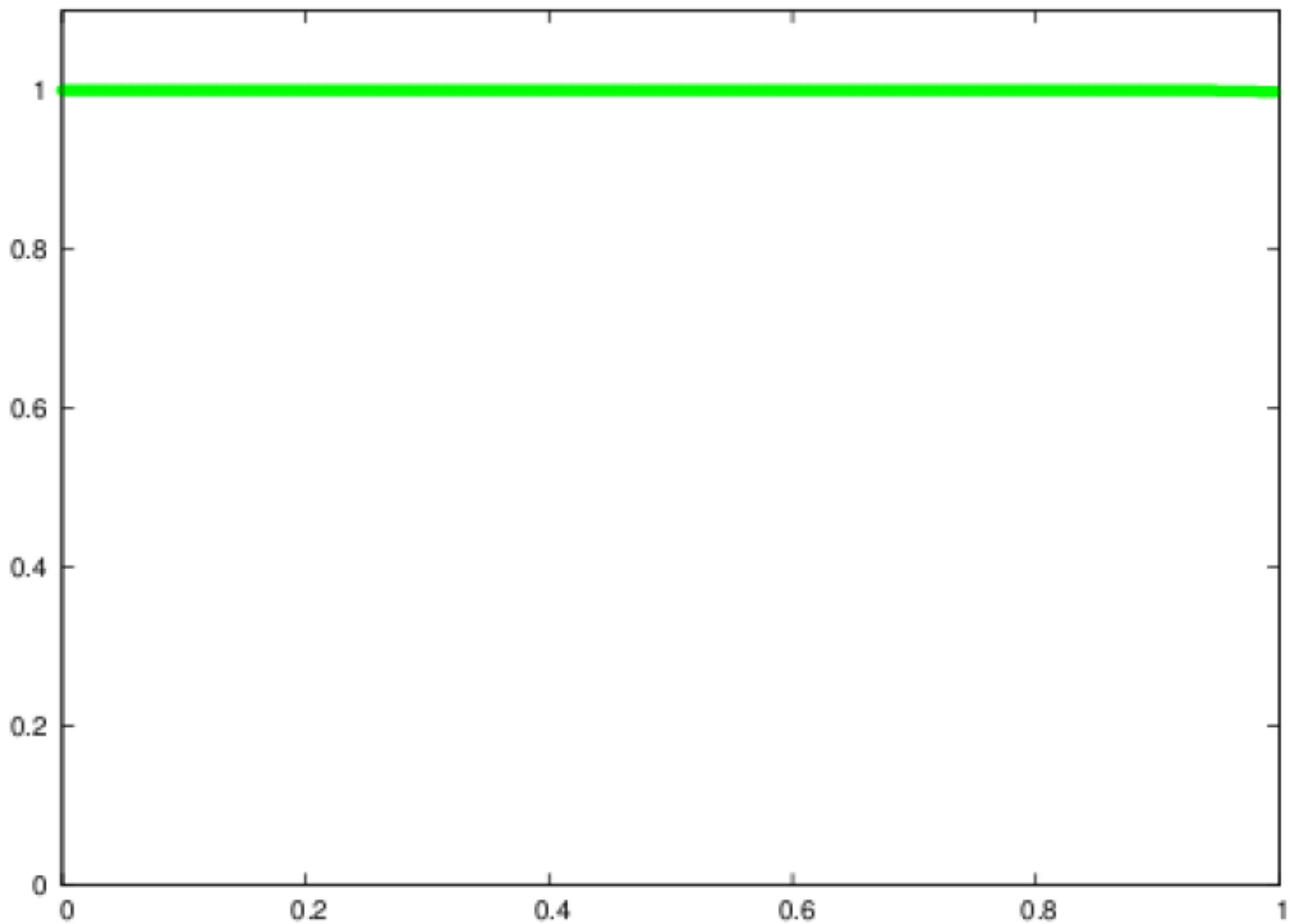


+
Nonrigid DB – Zoom 1



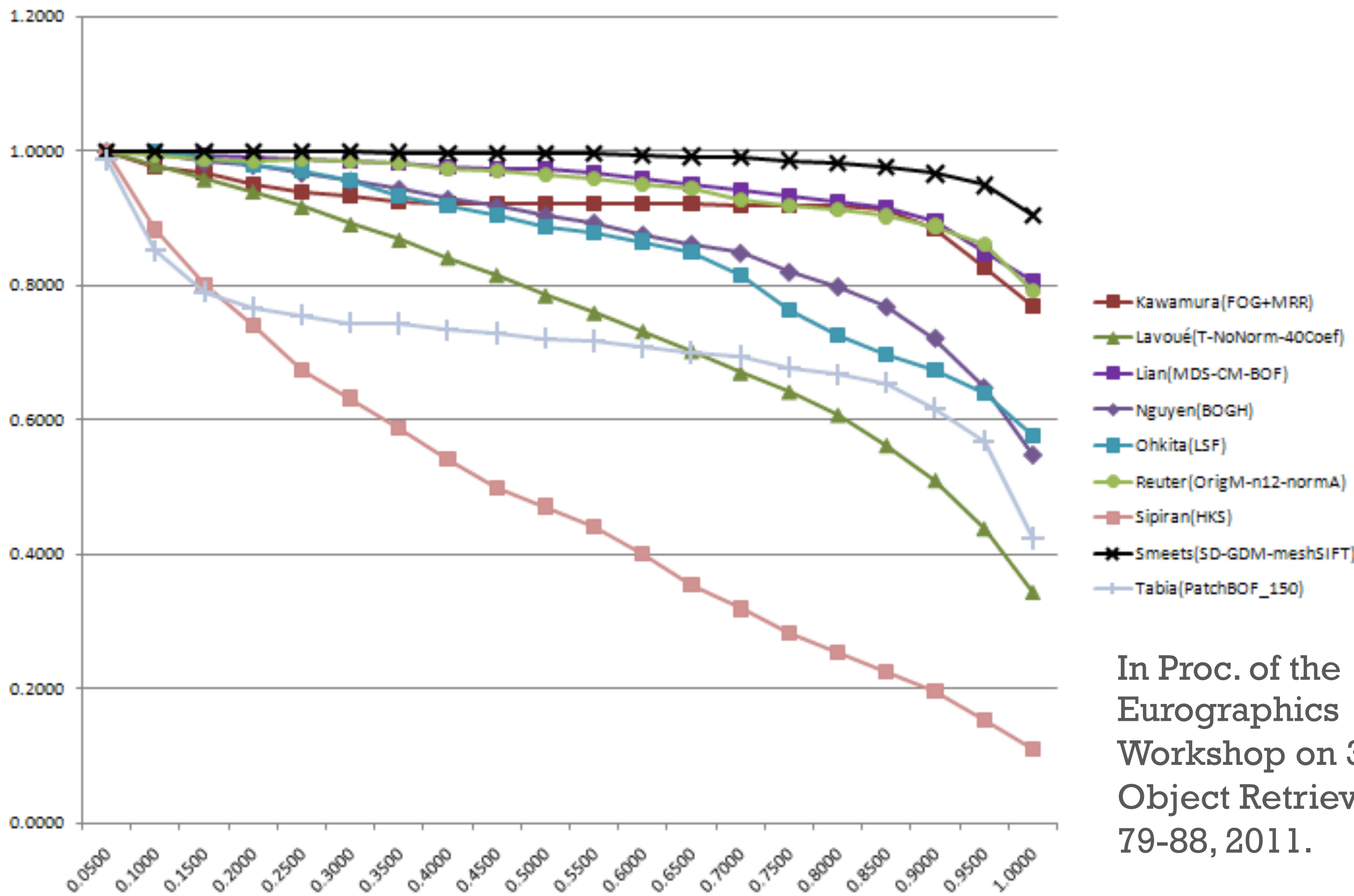
Nonrigid DB

$P(R) := Precision(Recall)$ averaged:



Integral: 0.999947 and min: 0.99829

Shape Retrieval Contest 11 Non Rigid Track

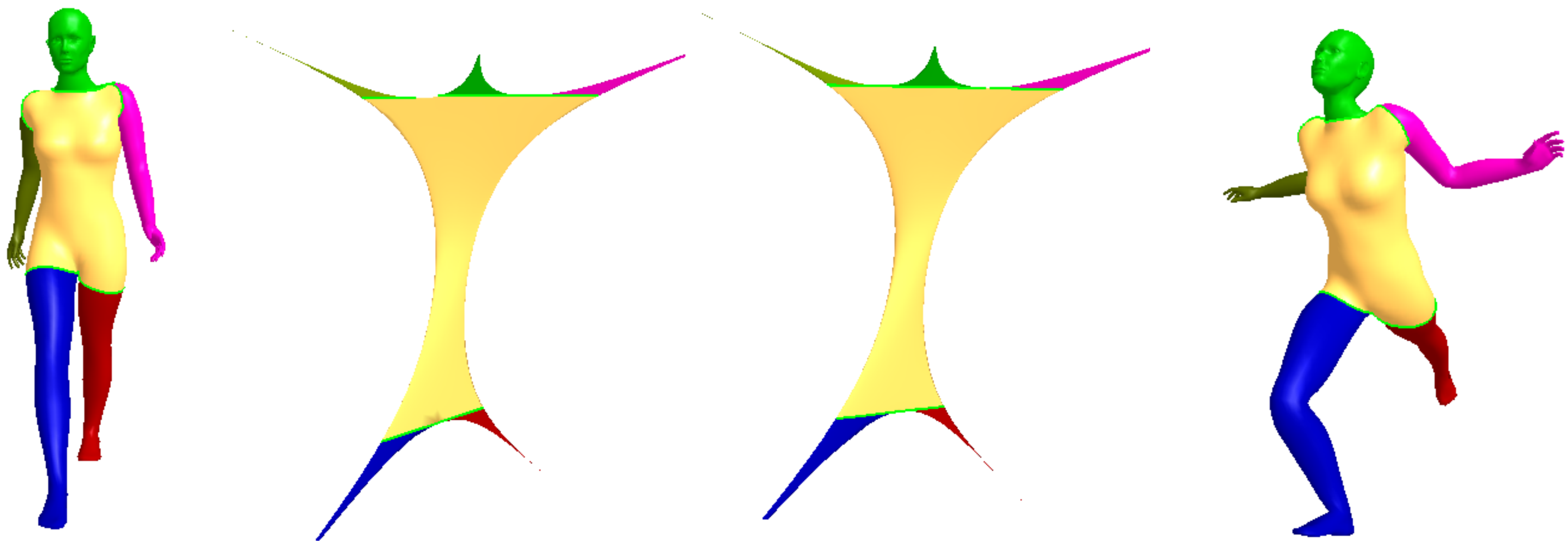


In Proc. of the Eurographics Workshop on 3D Object Retrieval, pp. 79-88, 2011.

Spectral Embedding

- Maps each vertex to the value of all (or a few) Eigenfunctions at that location:

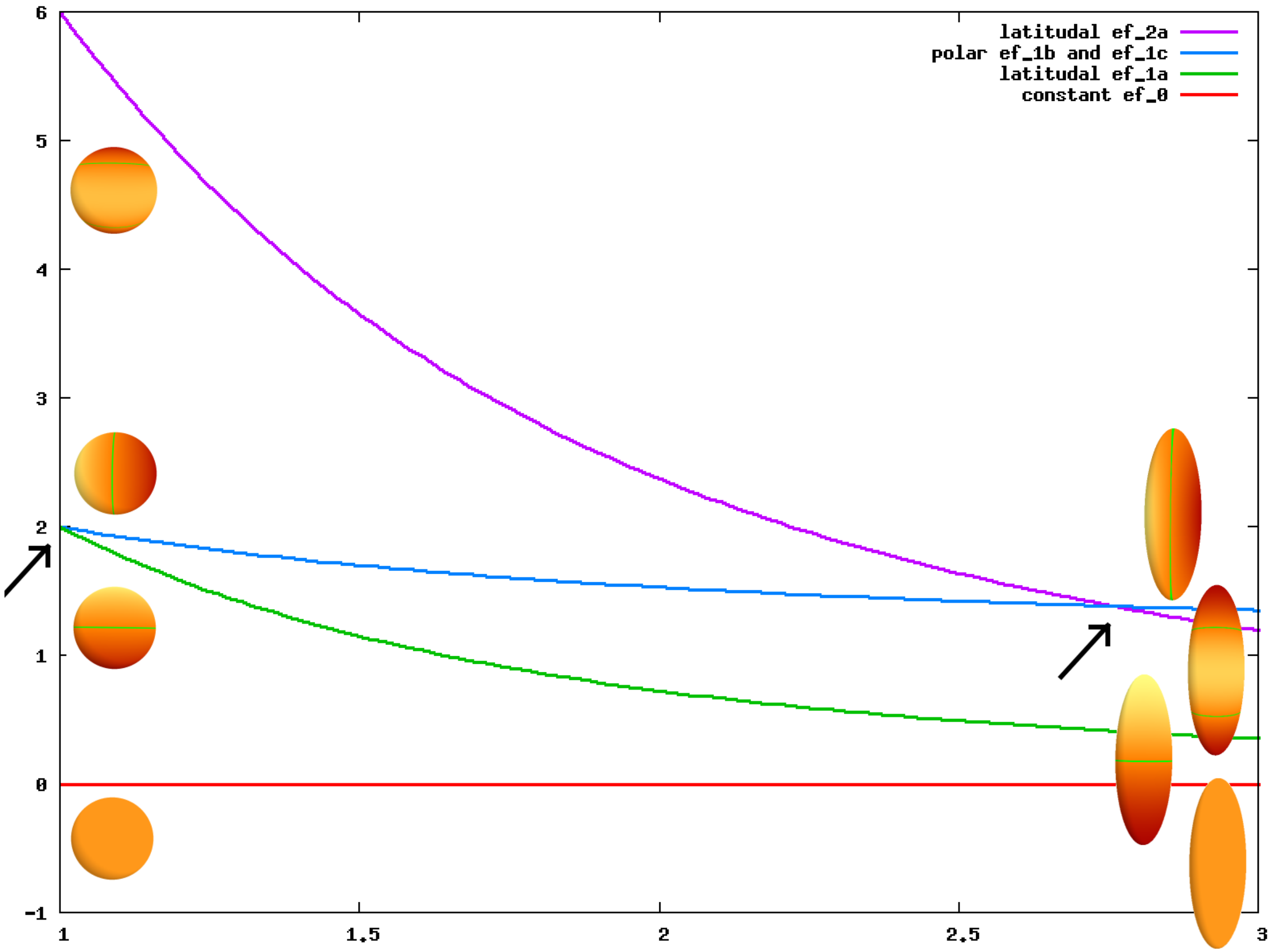
$$\Phi(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}), ...)$$



Difficulties

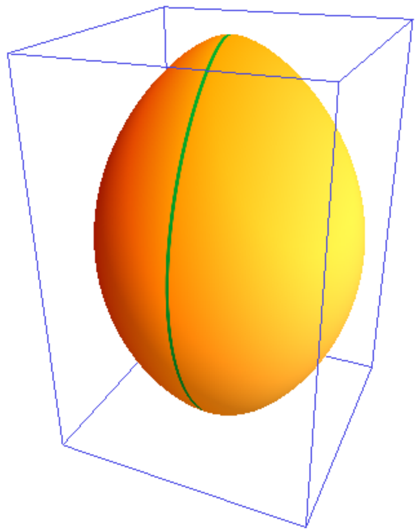
- Numerical inaccuracies
- Sign Flips
  - Scalar multiples are contained in same space
- Higher dimensional Eigenspaces
  - Arbitrary basis (luckily they are rare, but being close to one is already problematic)
  - Due to numerical errors Eigenvalues will be slightly different and we cannot really detect these situations
- Switching of Eigenfunctions
  - Occurs, because of numerical instabilities (two close Eigenvalues switch)
  - Or due to geometric (non-isometric) deformations

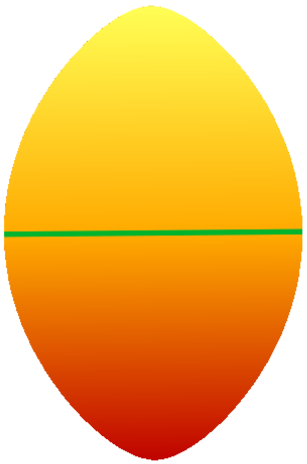
Switching of Eigenfunctions



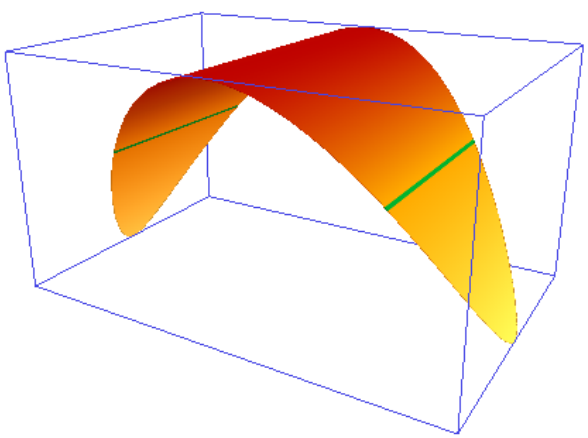
Switching of Eigenfunctions

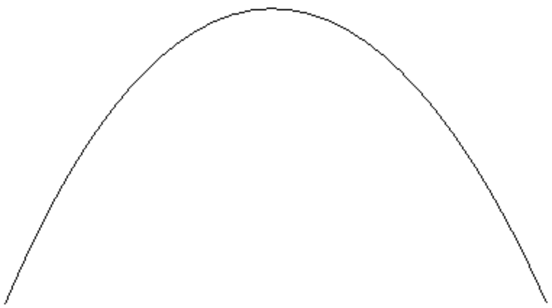
- Z=2.74





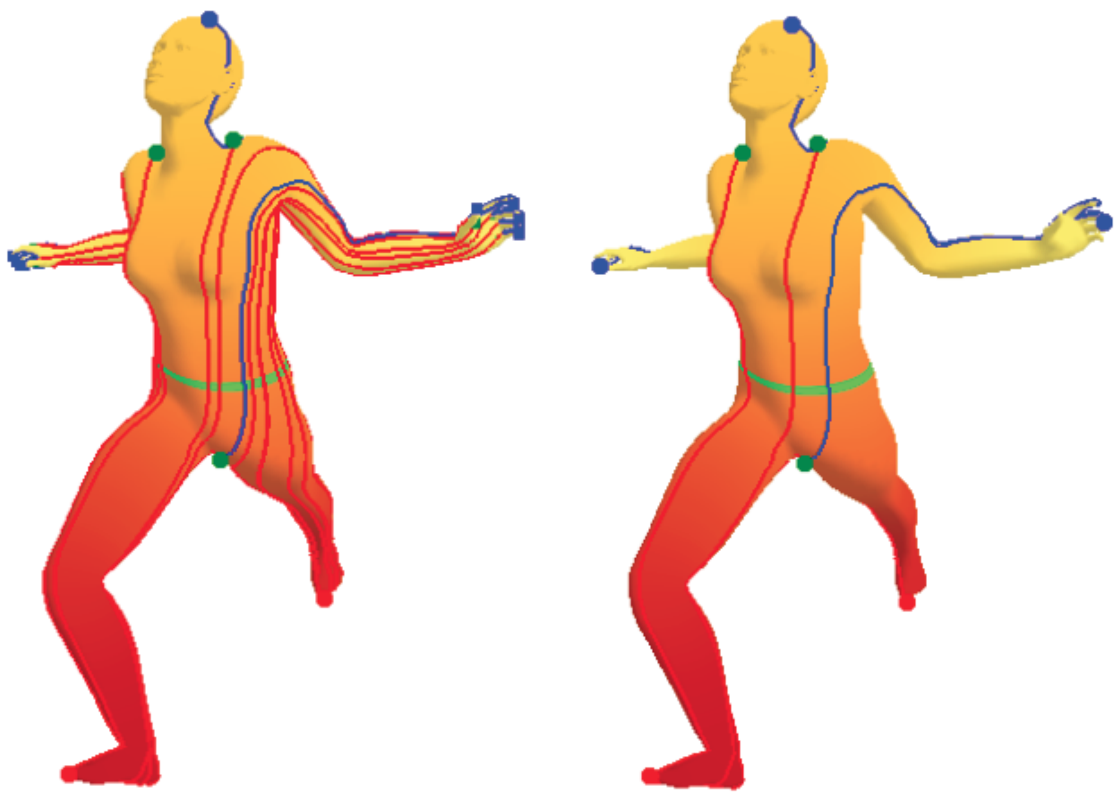
- Z=2.76





Shape Segmentation

- Morse-Smale Complex of the 1st Eigenfunction
  - Left: full complex Right: simplified (3min,2max,3saddles)



[IJCV09]

Shape Segmentation

- Segmentation on different 'persistence' levels
  - Left: using only the most significant critical points
  - Right: close-up of hand using all (except noise)

[IJCV09]

Hierarchical Segmentation

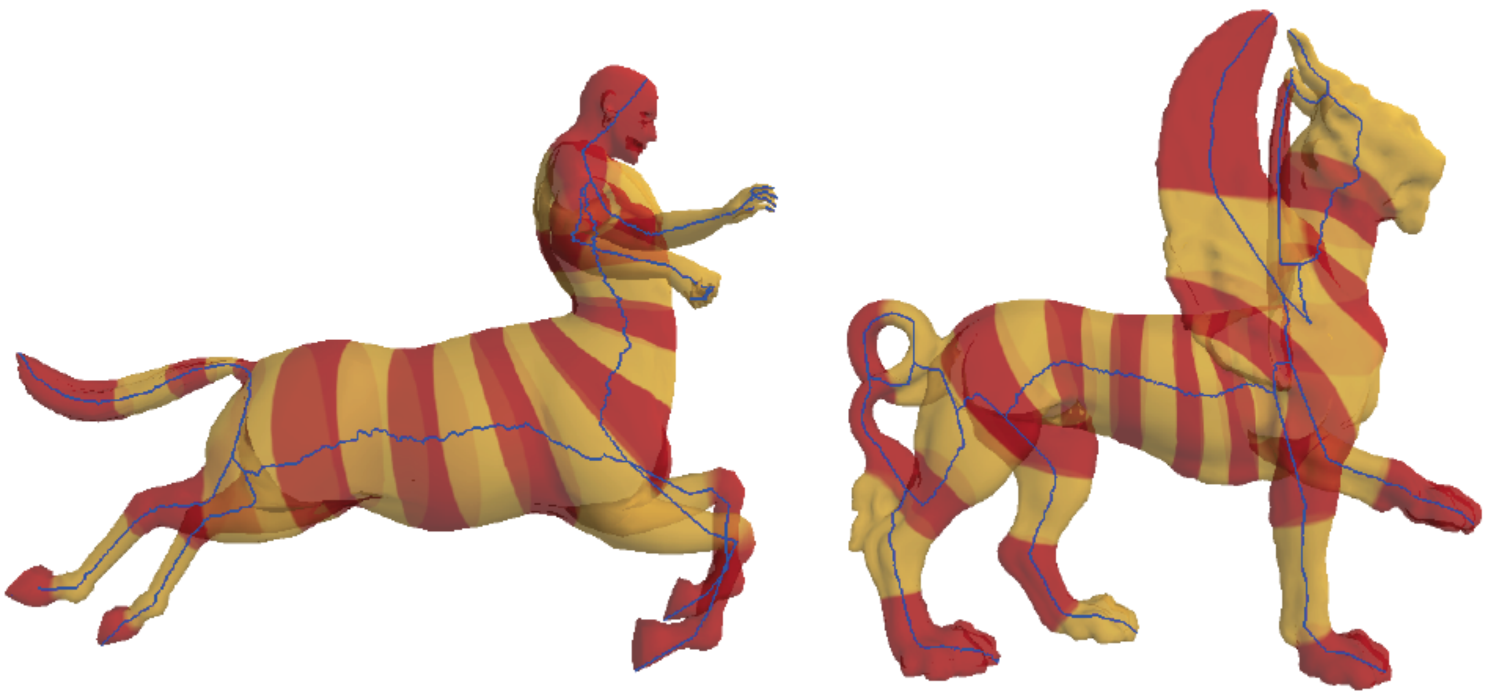
[IJCV09]

Consistent Segmentation and Registration

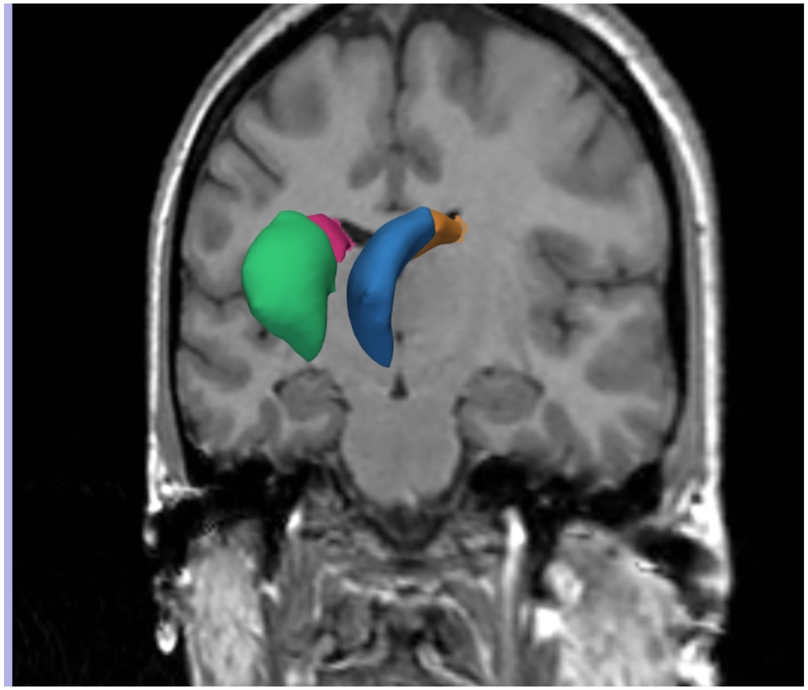
[IJCV09]

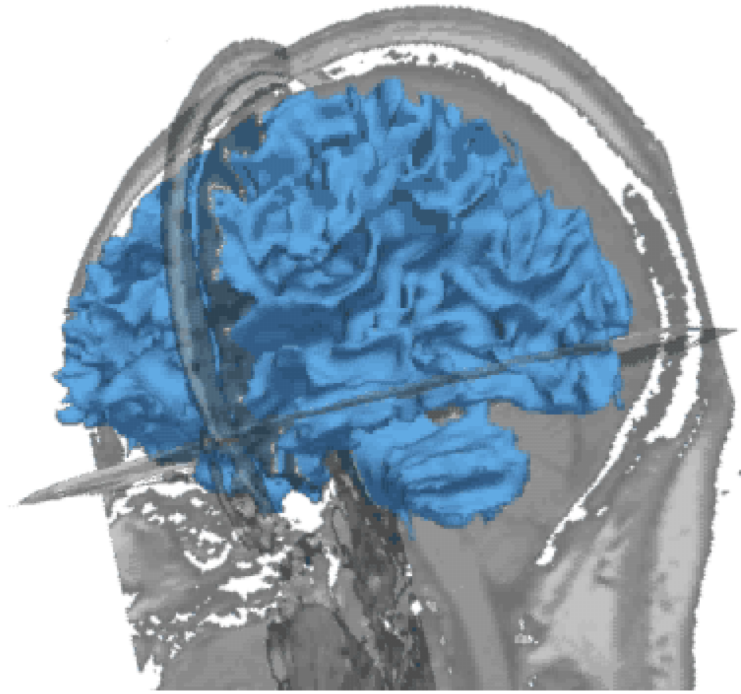
Other Applications

- Dense correspondence: Texture or Marker transfer, Surgical Planning
- Segmentation plus Skeleton: Pose Interpolation, Animation



Caudate Nucleus

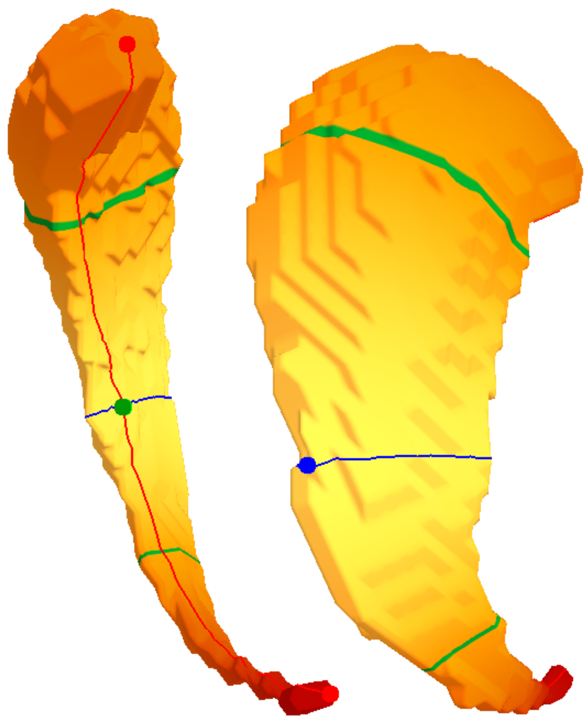




- Involved in memory function, emotion processing, and learning
  - Psychiatry Neuroimaging Lab (BWH - Martha Shenton)
  - Population: 32 Schizotypal Personality Disorder, 29 NC

[MICCAI07], [CW08], [CAD09]

Shape Analysis Caudate

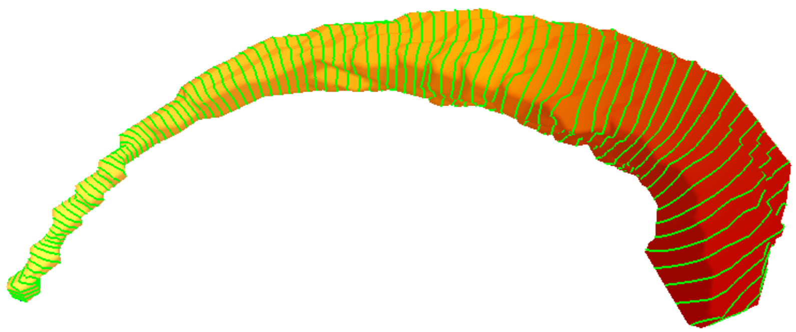


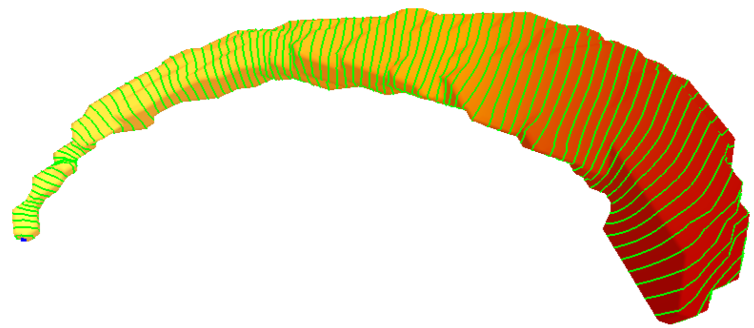
- Eigenfunction (EF): 2
- maxima at tips (red)
- minimum at outer rim (blue, middle)
- saddle at inner rim (green, left),
- integral lines (red and blue curve) run from the saddle to the extrema
- closed green curves denote the zero level sets

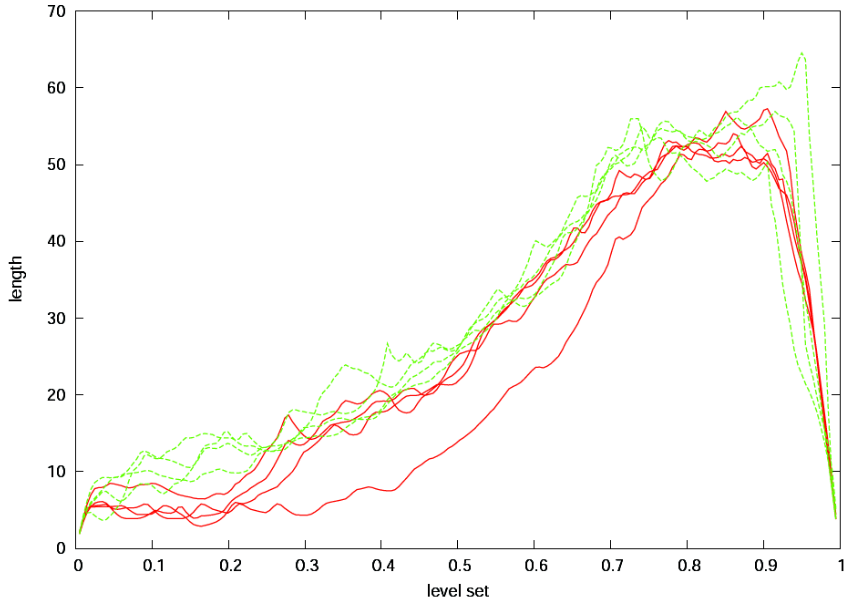
- (h) the head circumference (long green curve)
- (w) the waist circumference (blue curve)
- (t) the tail circumference (short green curve)
- (l) the length (red curve).

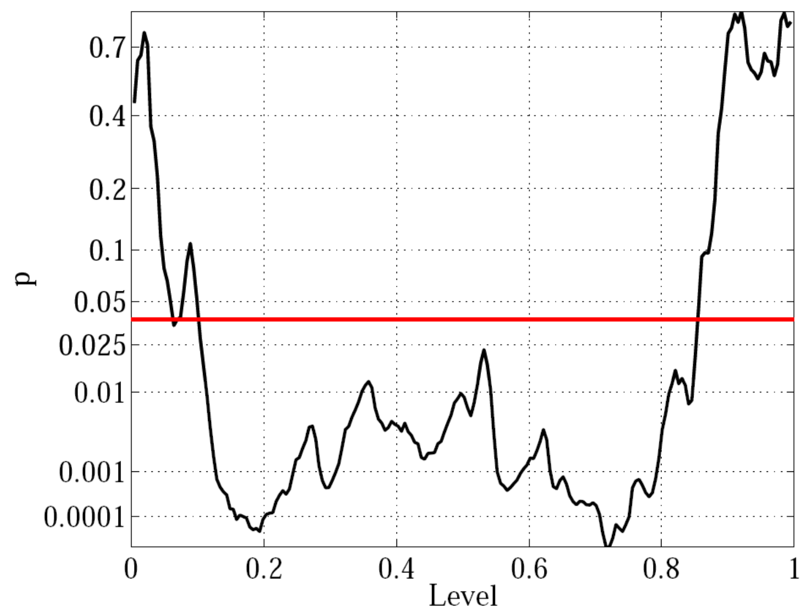
[MICCAI07], [CW08], [CAD09]

Shape Analysis Caudate



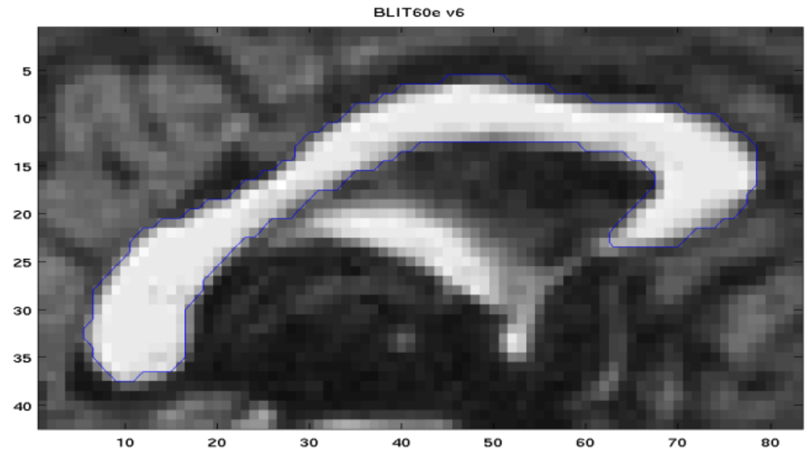


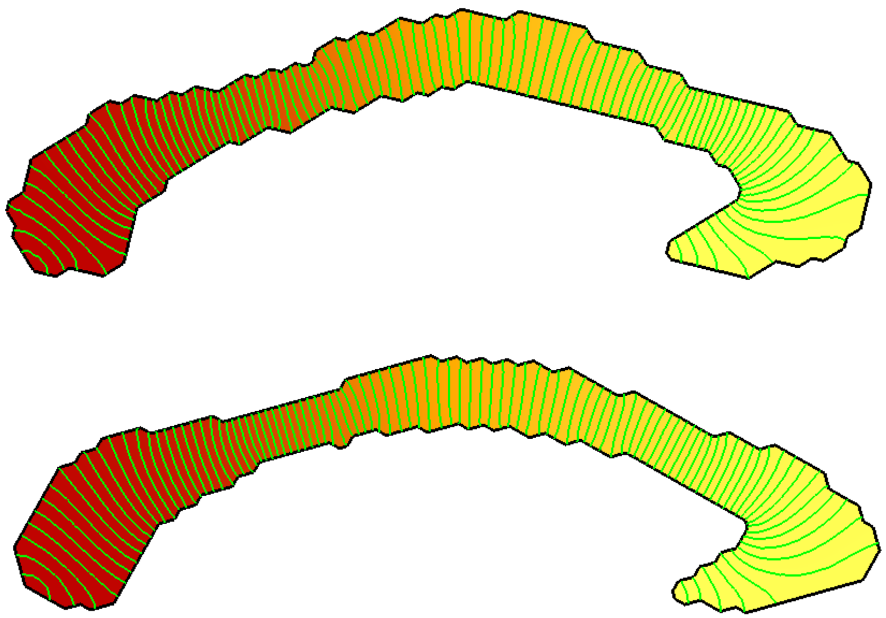


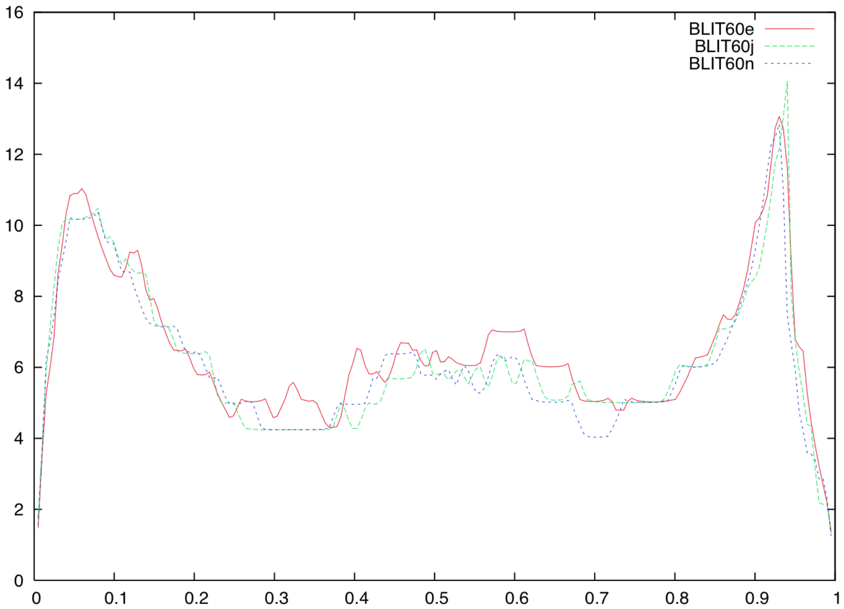


[MICCAI07], [CW08], [CAD09]

Shape Analysis Corpus Callosum







Thanks

Publications and Software:
http://reuter.mit.edu