A Natural Extension of Laplacian Eigenfunctions from Interior to Exterior and its Application

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Outline

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Acknowledgment

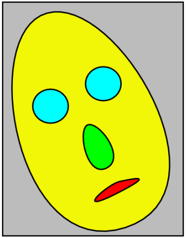
- Mark Ashbaugh (Univ. Missouri)
- John Hunter (UC Davis)
- ONR

Outline

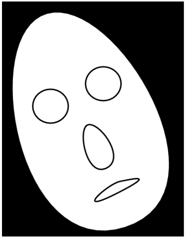
Motivations

- Consider a bounded domain of general (may be quite complicated) shape $\Omega \subset \mathbb{R}^d$.
- Want to analyze the spatial frequency information inside of the object defined in $\Omega \Longrightarrow$ need to avoid the Gibbs phenomenon due to $\partial\Omega$.
- Want to represent the object information efficiently for analysis, interpretation, discrimination, etc. $\Longrightarrow$ fast decaying expansion coefficients relative to a meaningful basis.
- Want to extract geometric information about the domain $\Omega \Longrightarrow$ shape clustering/classification.

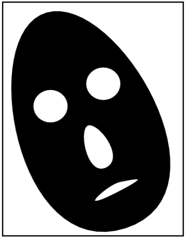
Motivations . . . Object-Oriented Image Analysis



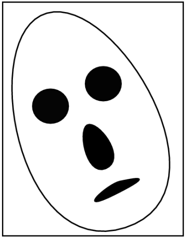
(a) Original



(b) Background

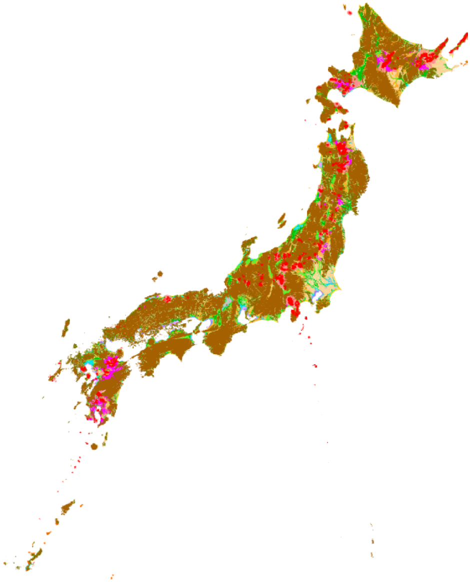


(c) Object



(d) Anomalies

Motivations . . . Data Analysis on a Complicated Domain



Outline

Eigenfunctions of Laplacian

- Why not analyze (and synthesize) the object (or region) of interest using genuine basis functions tailored to the domain?
- After all, *sines* (and *cosines*) are the eigenfunctions of the Laplacian on the *rectangular* domain with Dirichlet (and Neumann) boundary condition.
- *Spherical harmonics, Bessel functions, and Prolate Spheroidal Wave Functions*, are part of the eigenfunctions of the Laplacian (via separation of variables) for the *spherical, cylindrical, and spheroidal* domains, respectively.

Eigenfunctions of Laplacian . . .

- Consider an operator $\mathscr{L} = -\Delta$ in $L^2(\Omega)$ with *appropriate* boundary condition.
- Dealing with $\mathscr{L}$ is difficult due to unboundedness, etc.
- Much better to deal with its inverse, i.e., the Green's operator because it is compact and self-adjoint.
- Thus $\mathscr{L}^{-1}$ has discrete spectra (i.e., a countable number of eigenvalues with finite multiplicity) except 0 spectrum.
- $\mathscr{L}$ has a complete orthonormal basis of $L^2(\Omega)$, and this allows us to do eigenfunction expansion in $L^2(\Omega)$.

Eigenfunctions of Laplacian . . . Difficulties

- The key difficulty is to compute such eigenfunctions; directly solving the Helmholtz equation (or eigenvalue problem) on a general domain is tough.
- Unfortunately, computing the Green's function for a general Ω satisfying the usual boundary condition (i.e., Dirichlet, Neumann) is also very difficult.

Outline

Integral Operators Commuting with Laplacian

- The key idea is to find an integral operator commuting with the Laplacian without imposing the strict boundary condition a priori.
- Then, we know that the eigenfunctions of the Laplacian is the same as those of the integral operator, which is easier to deal with, due to the following

Theorem (G. Frobenius 1896?; B. Friedman 1956)

Suppose $\mathscr{K}$ and $\mathscr{L}$ commute and one of them has an eigenvalue with finite multiplicity. Then, $\mathscr{K}$ and $\mathscr{L}$ share the same eigenfunction corresponding to that eigenvalue. That is, $\mathscr{L}\varphi = \lambda\varphi$ and $\mathscr{K}\varphi = \mu\varphi$.

Integral Operators Commuting with Laplacian ...

- Let's replace the Green's function $G(\boldsymbol{x},\boldsymbol{y})$ by the fundamental solution of the Laplacian:

$$K(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} -\frac{1}{2}|x-y| & \text{if } d=1, \\ -\frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{y}| & \text{if } d=2, \\ \frac{|\boldsymbol{x}-\boldsymbol{y}|^{2-d}}{(d-2)\omega_d} & \text{if } d>2, \end{cases}$$

  where $\omega_d := \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the unit ball in $\mathbb{R}^d$, and $|\cdot|$ is the standard Euclidean norm.
- The price we pay is to have rather implicit, non-local boundary condition although we do not have to deal with this condition directly.

- Let $\mathscr{K}$ be the integral operator with its kernel $K(\boldsymbol{x},\boldsymbol{y})$:

$$\mathscr{K} f(\boldsymbol{x}) := \int_{\Omega} K(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad f \in L^2(\Omega).$$

Theorem (NS 2005)

The integral operator $\mathscr{K}$ commutes with the Laplacian $\mathscr{L} = -\Delta$ with the following non-local boundary condition:

$$\int_{\partial\Omega} K(\boldsymbol{x},\boldsymbol{y}) \frac{\partial \varphi}{\partial \nu_{\boldsymbol{y}}}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}) = -\frac{1}{2}\varphi(\boldsymbol{x}) + \mathrm{pv}\int_{\partial\Omega} \frac{\partial K(\boldsymbol{x},\boldsymbol{y})}{\partial \nu_{\boldsymbol{y}}} \varphi(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}),$$

for all $\boldsymbol{x} \in \partial\Omega$, where φ is an eigenfunction common for both operators.

Corollary (NS 2009)

The eigenfunction $\varphi(\boldsymbol{x})$ of the integral operator $\mathcal{K}$ in the previous theorem can be extended outside the domain Ω and satisfies the following equation:

$$-\Delta\varphi = \begin{cases} \lambda\varphi & \text{if } \boldsymbol{x} \in \Omega; \\ 0 & \text{if } \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega}, \end{cases}$$

with the boundary condition that φ and $\dfrac{\partial\varphi}{\partial\nu}$ are continuous across the boundary $\partial\Omega$. Moreover, as $|\boldsymbol{x}| \to \infty$, $\varphi(\boldsymbol{x})$ must be of the following form:

$$\varphi(\boldsymbol{x}) = \begin{cases} \mathrm{const}\cdot|\boldsymbol{x}|^{2-d} + O\left(|\boldsymbol{x}|^{1-d}\right) & \text{if } d \neq 2; \\ \mathrm{const}\cdot\ln|\boldsymbol{x}| + O\left(|\boldsymbol{x}|^{-1}\right) & \text{if } d = 2. \end{cases}$$

Outline

1D Example

- Consider the unit interval $\Omega = (0,1)$.
- Then, our integral operator $\mathscr{K}$ with the kernel $K(x,y) = -|x-y|/2$ gives rise to the following eigenvalue problem:

$$-\varphi'' = \lambda\varphi, \quad x \in (0,1);$$

$$-\varphi'(0) = \varphi'(1) = \varphi(0) + \varphi(1).$$

- The kernel $K(x,y)$ is of Toeplitz form $\Longrightarrow$ Eigenvectors must have even and odd symmetry (Cantoni-Butler '76).
- In this case, we have the following explicit solution.

1D Example ...

- $\lambda_0 \approx -5.756915$, which is a solution of $\tanh \frac{\sqrt{-\lambda_0}}{2} = \frac{2}{\sqrt{-\lambda_0}}$,

$$\varphi_0(x) = A_0 \cosh \sqrt{-\lambda_0}\left(x - \frac{1}{2}\right);$$

- $\lambda_{2m-1} = (2m-1)^2\pi^2$, $m = 1, 2, \ldots$,

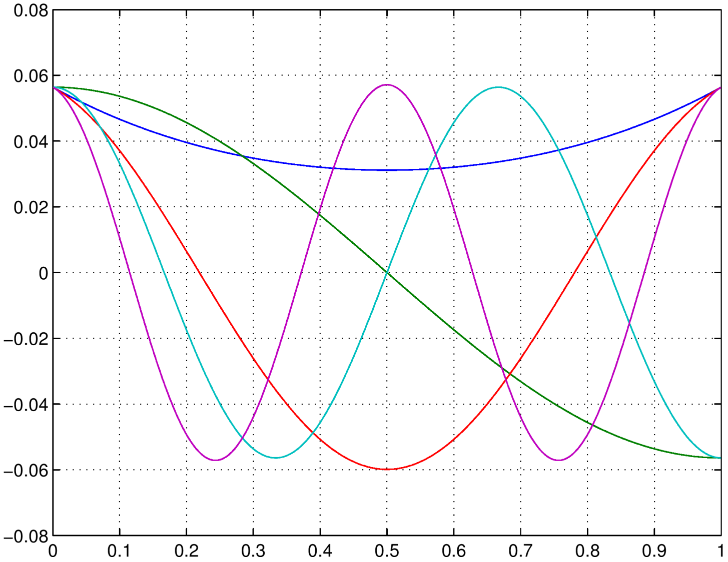
$$\varphi_{2m-1}(x) = \sqrt{2}\cos(2m-1)\pi x;$$

- λ_{2m}, $m = 1, 2, \ldots$, which are solutions of $\tan \frac{\sqrt{\lambda_{2m}}}{2} = -\frac{2}{\sqrt{\lambda_{2m}}}$,

$$\varphi_{2m}(x) = A_{2m} \cos \sqrt{\lambda_{2m}}\left(x - \frac{1}{2}\right),$$

where A_k, $k = 0, 1, \ldots$ are normalization constants.

First 5 Basis Functions



2D Example

- Consider the unit disk Ω. Then, our integral operator $\mathscr{K}$ with the kernel $K(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{2\pi} \log|\boldsymbol{x} - \boldsymbol{y}|$ gives rise to:

$$-\Delta\varphi = \lambda\varphi, \quad \text{in } \Omega;$$

$$\left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = \left.\frac{\partial\varphi}{\partial r}\right|_{\partial\Omega} = -\left.\frac{\partial\mathscr{H}\varphi}{\partial\theta}\right|_{\partial\Omega}$$

where $\mathscr{H}$ is the Hilbert transform for the circle, i.e.,

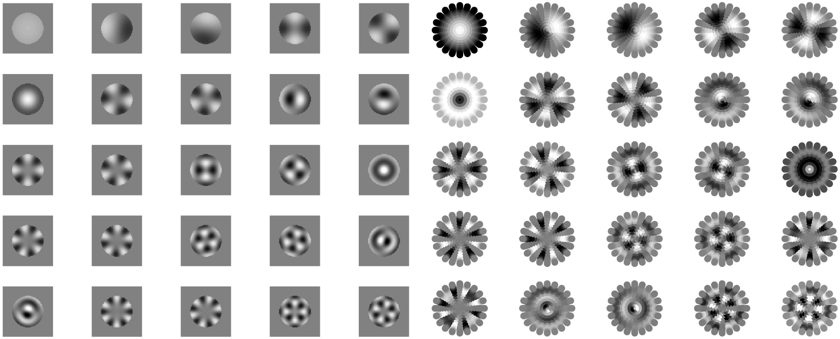
$$\mathscr{H}f(\theta) := \frac{1}{2\pi}\,\mathrm{pv}\int_{-\pi}^{\pi} f(\eta)\cot\left(\frac{\theta-\eta}{2}\right)\mathrm{d}\eta \quad \theta \in [-\pi, \pi].$$

- Let $\beta_{k,\ell}$ is the ℓth zero of the Bessel function of order k, $J_k(\beta_{k,\ell}) = 0$. Then,

$$\varphi_{m,n}(r,\theta) = \begin{cases} J_m(\beta_{m-1,n}\, r)\binom{\cos}{\sin}(m\theta) & \text{if } m = 1, 2, \ldots,\ n = 1, 2, \ldots, \\ J_0(\beta_{0,n}\, r) & \text{if } m = 0,\ n = 1, 2, \ldots, \end{cases}$$

$$\lambda_{m,n} = \begin{cases} \beta^2_{m-1,n}, & \text{if } m = 1, \ldots,\ n = 1, 2, \ldots, \\ \beta^2_{0,n} & \text{if } m = 0,\ n = 1, 2, \ldots. \end{cases}$$

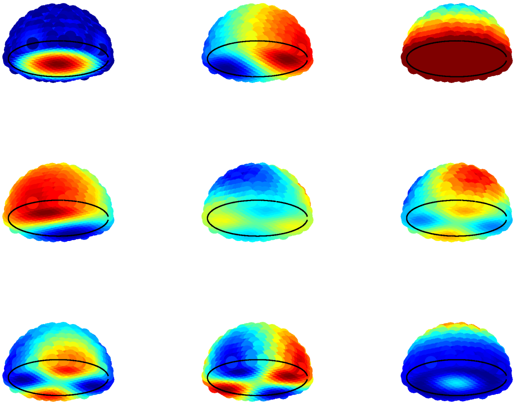
First 25 Basis Functions



(a) Our Basis (b) Dirichlet-Laplacian Basis

3D Example

- Consider the unit ball Ω in $\mathbb{R}^3$. Then, our integral operator $\mathscr{K}$ with the kernel $K(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$.
- Top 9 eigenfunctions cut at the equator viewed from the south:



Outline

Connection with von Neumann–Kreĭn Extension Theory

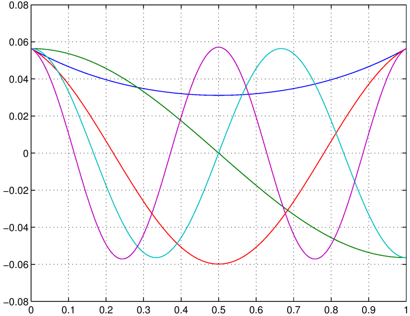
- John von Neumann (1929) and Mark Kreĭn (1947) considered a self-adjoint extension of symmetric operators.
- Let $T := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\mathscr{D}(T) := H_0^2(0,1) \subset H^2(0,1)$, where $H_0^2(0,1) := \{f \in H^2(0,1) \mid f(0) = f(1) = f'(0) = f'(1) = 0\}$ and $H^2(0,1) := \{f \in C^1[0,1] \mid f' \in AC[0,1], f'' \in L^2(0,1)\}$. T is a positive *symmetric* operator on $\mathscr{D}(T)$, but *not self-adjoint* because $\mathscr{D}(T^*) = H^2(0,1) \supsetneq \mathscr{D}(T)$.
- von Neumann-Kreĭn extension of T is the smallest (or soft) self-adjoint extension $T_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\mathscr{D}(T_0) = \{f \in H^2(0,1) \mid f'(0) = f'(1) = f(1) - f(0)\} = \mathscr{D}(T_0^*)$.
- Compare it with our boundary condition: $-f'(0) = f'(1) = f(0) + f(1)$.
- Also, compare it with the *Friedrichs extension* of T, which is the *largest (or hard) self-adjoint extension*: $T_\infty = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\mathscr{D}(T_\infty) = \{f \in H^2(0,1) \mid f(0) = f(1) = 0\} = \mathscr{D}(T_\infty^*) \Longleftrightarrow$ Dirichlet BC!

Connection with von Neumann–Kreĭn Extension Theory ...

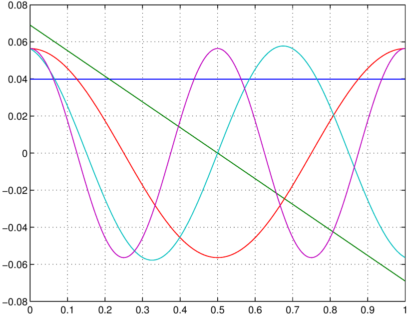
	Our Basis	Kreĭn-Laplacian Basis
λ_0	-5.756915...; $\tanh\sqrt{-\lambda_0}/2 = 2/\sqrt{-\lambda_0}$	0
φ_0	$\cosh\sqrt{-\lambda_0}(x-1/2)$	1
λ_{2m-1}	$((2m-1)\pi)^2$	$\tan\sqrt{\lambda_{2m-1}}/2 = \sqrt{\lambda_{2m-1}}/2$
φ_{2m-1}	$\sin(2m-1)\pi(x-1/2)$	$\sin\sqrt{\lambda_{2m-1}}(x-1/2)$
λ_{2m}	$\tan\sqrt{\lambda_{2m}}/2 = -2/\sqrt{\lambda_{2m}}$	$(2m\pi)^2$
φ_{2m}	$\cos\sqrt{\lambda_{2m}}(x-1/2)$	$\cos 2m\pi(x-1/2)$

Note that the above eigenfunctions are not normalized to have $\|\cdot\|_2 = 1$.

Connection with von Neumann–Kreĭn Extension Theory . . .



(a) Our Basis



(b) Kreĭn-Laplacian Basis

Connection with von Neumann–Kreĭn Extension Theory . . .

- In higher dimensions, the von Neumann-Kreĭn extension of the Laplacian $T = -\Delta$, $\mathscr{D}(T) = H_0^2(\Omega)$, on $\Omega \subset \mathbb{R}^d$ turns out to be: $T_0 = -\Delta$, $\mathscr{D}(T_0) = \left\{ f \in H^2(\Omega) \,\middle|\, \frac{\partial f}{\partial \nu}(\boldsymbol{x}) = \frac{\partial H(f)}{\partial \nu}(\boldsymbol{x}), \boldsymbol{x} \in \partial\Omega \right\}$ where $H(f)$ is a harmonic function in Ω with the boundary condition: $H(f) = f$ on $\partial\Omega$; See e.g., A. Alonso & B. Simon: "The Birman-Kreĭn-Vishik theory of self-adjoint extensions of semibounded operators," *J. Operator Theory*, vol. 4, pp. 251–270, 1980.
- This is closely related to our Polyharmonic Local Sine Transform (PHLST): N. Saito & J.-F. Remy: "The polyharmonic local sine transform: A new tool for local image analysis and synthesis without edge effect," *Applied & Computational Harmonic Analysis*, vol. 20, no. 1, pp. 41-73, 2006.
- After all, the von Neumann-Kreĭn extensions do not deal with the exterior of the domain Ω while our approach based on the commuting integral operators allow us to extend our eigenfunctions very naturally to the exterior of Ω.

Outline

Signal Extrapolation

- Recall the definition of the eigenvalue problem:

$$\varphi(\boldsymbol{x}) = \frac{1}{\mu}\int_{\Omega} K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \quad \text{where } \boldsymbol{x} \text{ can be } \textit{any} \text{ point in } \mathbb{R}^d.$$

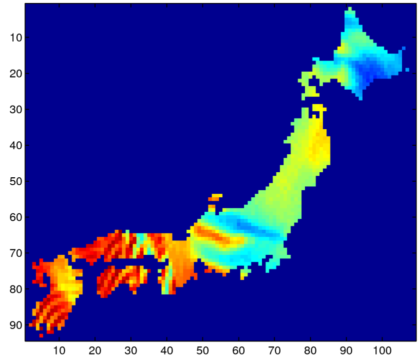
- For large μ (i.e., coarse scale/low frequency), extrapolation naturally extends to large area.
- For small μ (i.e., fine scale/high frequency), extrapolation quickly attenuates away from Ω.
- Now suppose we measure the expansion coefficients $\langle f, \varphi_k\rangle$ of a target function $f(\boldsymbol{x})$ on Ω and represent it by the eigenfunction expansion:

$$f(\boldsymbol{x}) = \sum_k \langle f, \varphi_k\rangle \varphi_k(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega.$$

- Using the above extension property of φ_k, we have a naturally extrapolation of f by

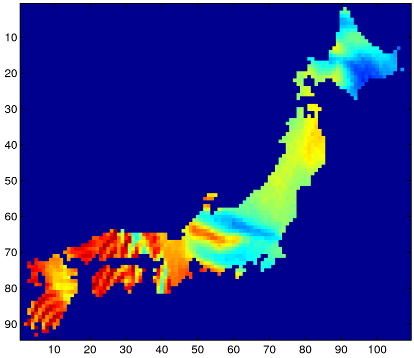
$$f(\boldsymbol{x}) = \sum_k \langle f, \varphi_k\rangle \varphi_k(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega}.$$

Signal Extrapolation: Example 1

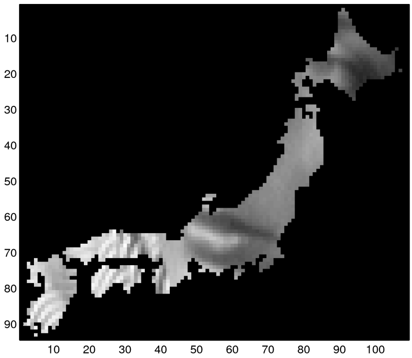


(a) What data?

Signal Extrapolation: Example 1

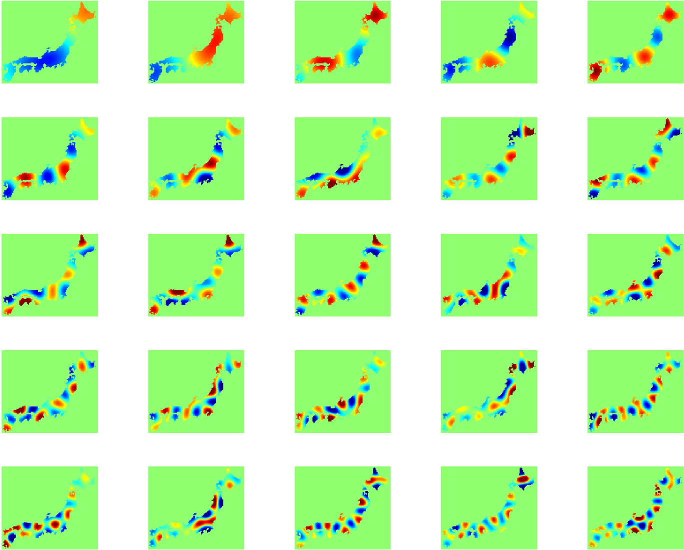


(a) What data?

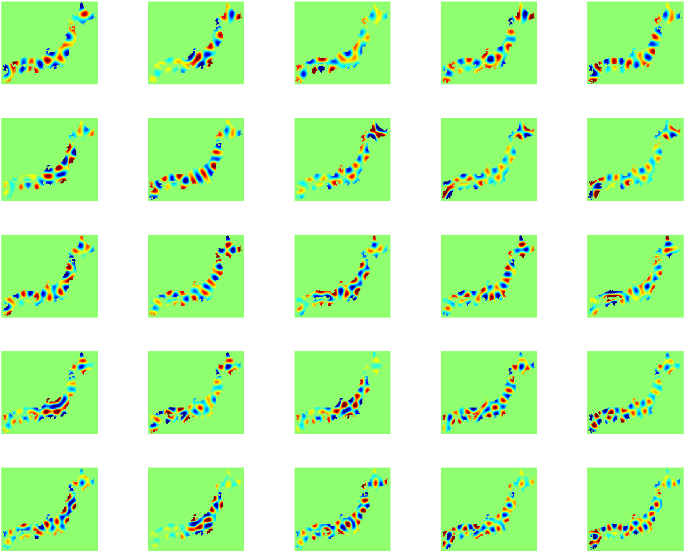


(b) $\chi_J \cdot$ Barbara

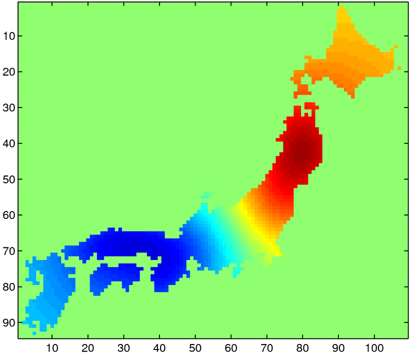
First 25 Basis Functions



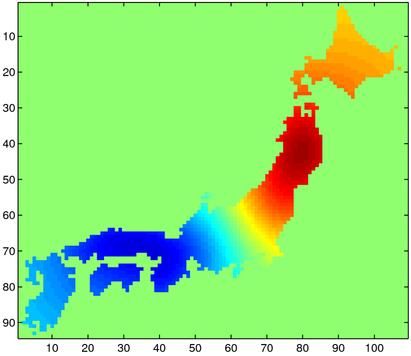
Next 25 Basis Functions



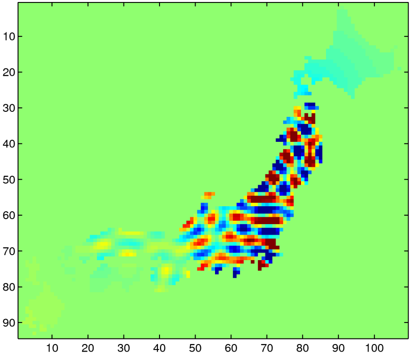
Extrapolated Basis Functions



(a) Low Freq. Extrap. from Honshu



(a) Low Freq. Extrap. from Honshu



(b) High Freq. Extrap. from Honshu

Extrapolated Data

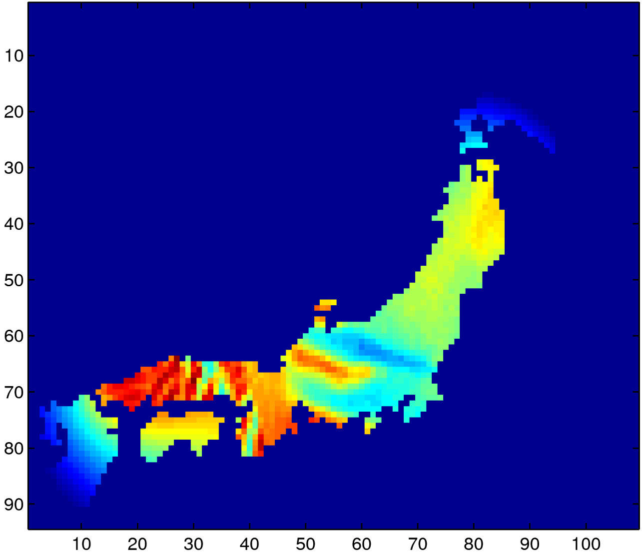
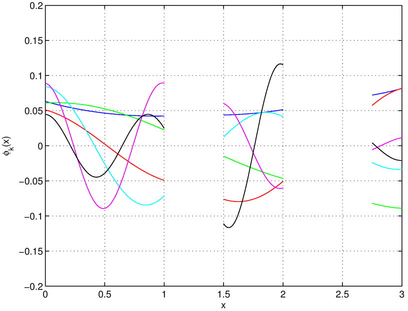


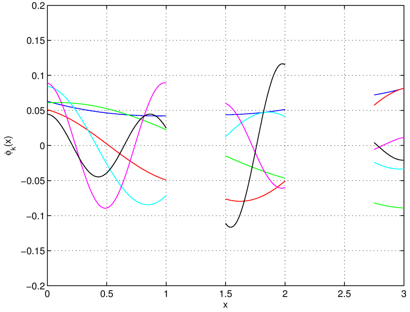
Figure : Extrapolation of χ_{Honshu}. Barbara to the three islands

Signal Extrapolation Example 2: Disconnected Intervals

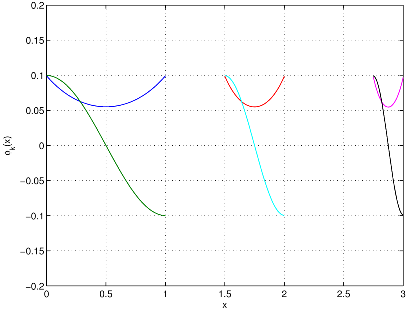


(a) United Basis (Top 6)

Signal Extrapolation Example 2: Disconnected Intervals

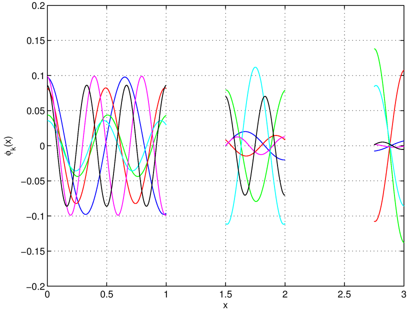


(a) United Basis (Top 6)



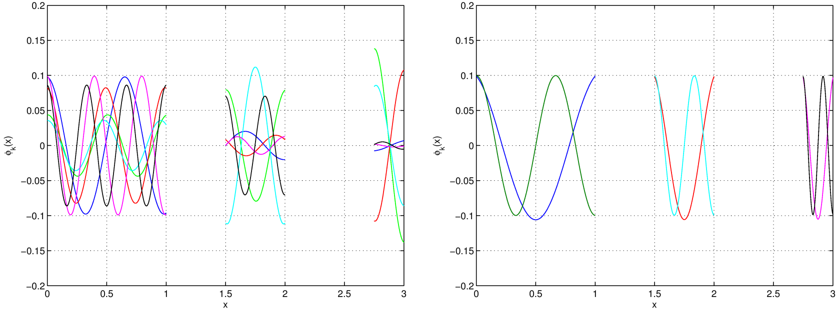
(b) Separated Bases (Top 2 in each subinterval)

Signal Extrapolation Example 2: Disconnected Intervals . . .



(a) United Basis (Next 6)

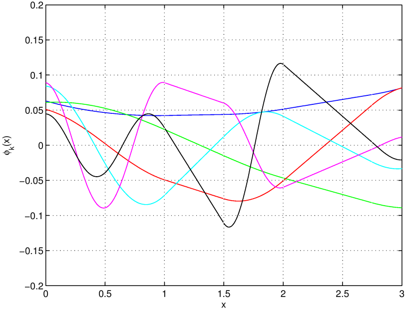
Signal Extrapolation Example 2: Disconnected Intervals . . .



(a) United Basis (Next 6)

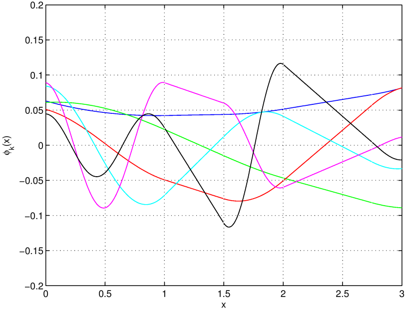
(b) Separated Bases (Next 2 in each subinterval)

Signal Extrapolation Example 2: Extrapolated Basis Vectors

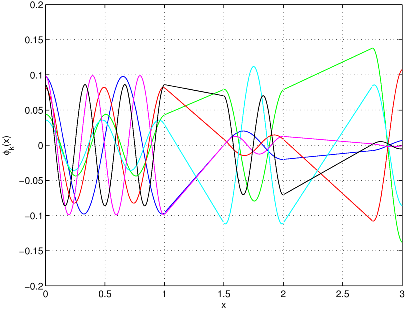


(a) United Basis (Top 6)

Signal Extrapolation Example 2: Extrapolated Basis Vectors

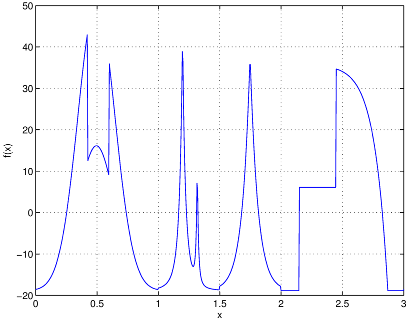


(a) United Basis (Top 6)



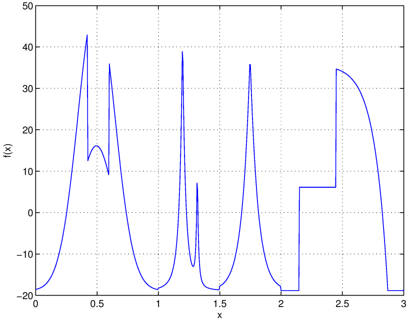
(b) United Bases (Next 6)

Signal Extrapolation Example 2: Approximation Experiments

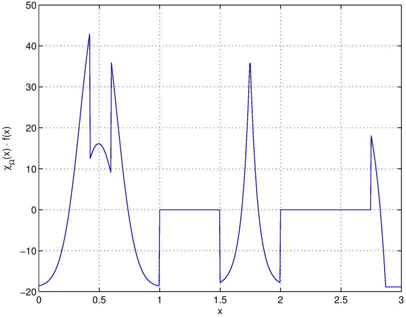


(a) Original Signal

Signal Extrapolation Example 2: Approximation Experiments

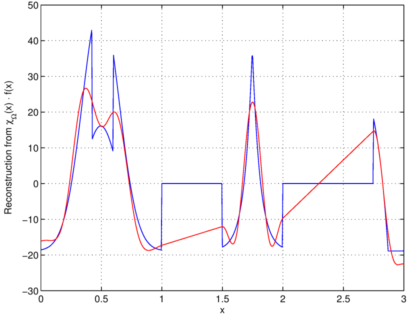


(a) Original Signal



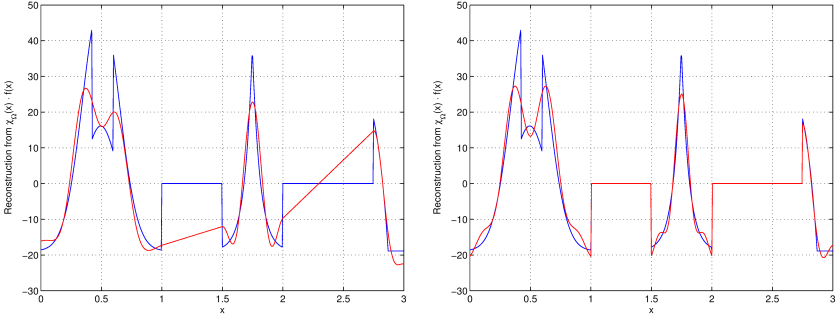
(b) Brutally Cut

Signal Extrapolation Example 2: Approximation Results



(a) Top 10 United Basis

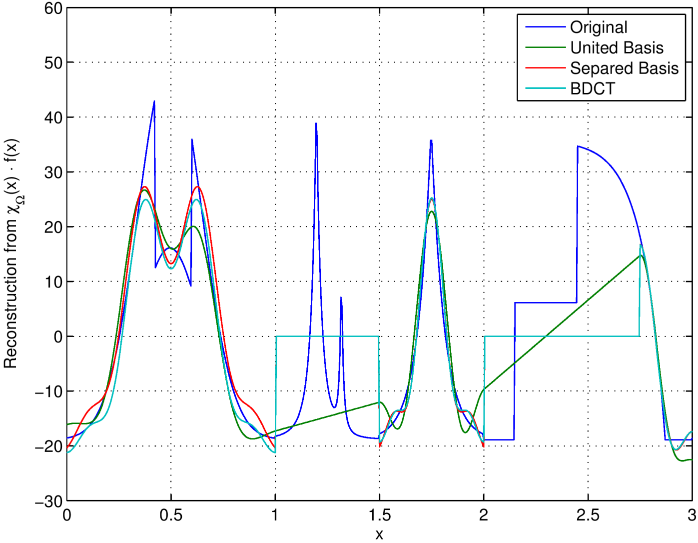
Signal Extrapolation Example 2: Approximation Results



(a) Top 10 United Basis

(b) Top 10 Separated Basis

Signal Extrapolation Example 2: Approximation Results



Outline

Conclusions

Our approach using the commuting integral operators

- Allows object-oriented signal/image analysis & synthesis
- Can get fast-decaying expansion coefficients (less Gibbs effect)
- Can naturally extend the basis functions outside of the initial domain
- Can extract geometric information of a domain through eigenvalues
- Can decouple geometry/domain information and statistics of data
- Is closely related to the von Neumann-Kreĭn Laplacian, yet is distinct
- Can use *Fast Multipole Methods* to speed up the computation, which is the key for higher dimensions/large domains
- Many things to be done:
  - Examine further our boundary conditions for specific geometry in higher dimensions; e.g., analysis of $\mathbb{S}^2$ leads to Clifford Analysis
  - Examine the relationship with the *Volterra operators* in $\mathbb{R}^d$, $d \geq 2$ (Lidskiĭ; Gohberg-Kreĭn)
  - Integral operators commuting with polyharmonic operators $(-\Delta)^p$, $p \geq 2$?

Outline

References

- Laplacian Eigenfunction Resource Page http://www.math.ucdavis.edu/~saito/lapeig/ contains
  - My Course Note (elementary) on "Laplacian Eigenfunctions: Theory, Applications, and Computations"
  - All the talk slides of the minisymposia on Laplacian Eigenfunctions at: ICIAM 2007, Zürich; SIAM Imaging Science Conference 2008, San Diego; IPAM 2009; and the other related recent minisymposia.
- The following articles are available at http://www.math.ucdavis.edu/~saito/publications/

  - N. Saito & J.-F. Remy: "The polyharmonic local sine transform: A new tool for local image analysis and synthesis without edge effect," *Applied & Computational Harmonic Analysis*, vol. 20, no. 1, pp. 41-73, 2006.
  - N. Saito: "Data analysis and representation using eigenfunctions of Laplacian on a general domain," *Applied & Computational Harmonic Analysis*, vol. 25, no. 1, pp. 68–97, 2008.

Thank you very much for your attention!