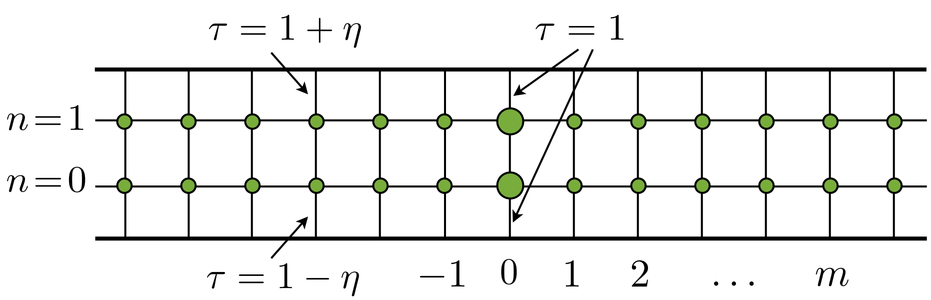
Embedded Eigenvalues of Coupled Graph Operators

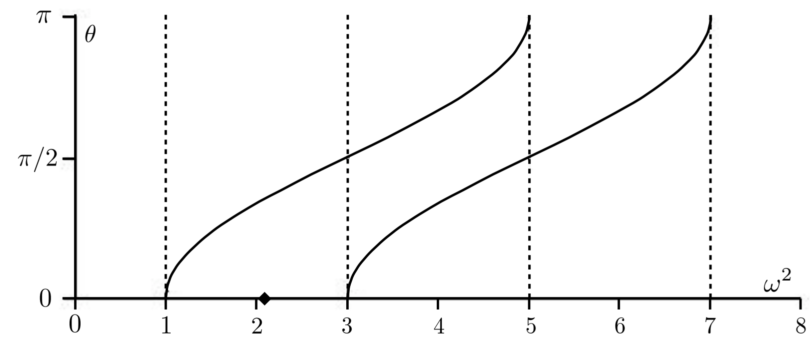
Stephen P. Shipman

Department of Mathematics
Louisiana State University
USA

Spectrally embedded eigenfunctions in 1D graph operators

1. Coupled biased chains





Floquet solutions $e^{i\theta m} q_n$ to $(Au) = -\omega^2 u$ $\rightsquigarrow$ Dispersion relation

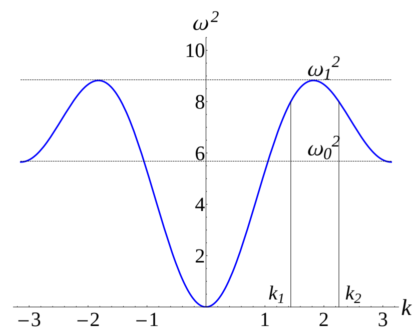
2. Next-nearest neighbor interactions $\rightsquigarrow$ bound state at embedded frequency

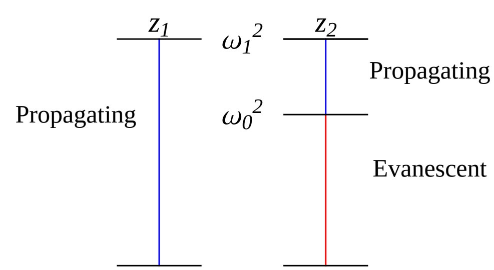
$$(Au)_n = \tau_2(u_{n+2} + u_{n-2}) + \tau_1(u_{n+1} + u_{n-1}) - 4u_n$$

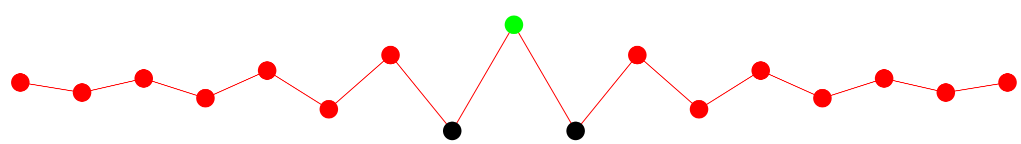
Floquet solutions

$$u_n = z^n = e^{ikn}$$

of $Au = -\omega^2 u$







Higher-D: Embedded eigenvalues of periodic graph operators

Given a n-periodic finite-order graph operator A, its **Floquet surface** at $\lambda \in \mathbb{C}$ is

$$\Psi_{A,\lambda} = \{ 0 \neq z \in \mathbb{C}^n \,:\, \det\left(\hat{A}(z) - \lambda\right) \neq 0 \}$$

$\hat{A}(z)$ is the multiplication-operator representation of A under the Floquet transform

$$u_n \mapsto \hat{u}_n(z) = \sum_{g \in \mathbb{Z}^n} u_{n+g} z^{-g}$$

The **spectrum** of A is

$$\sigma(A) = \left\{\lambda \,:\, \Psi_{A,\lambda} \cap \mathbb{T}^n \text{ is nonempty.}\right\}$$

Theorem. (Kuchment and Vainberg)
Let $\lambda \in \sigma(A)$, and let the Floquet surface $\Psi_{A,\lambda}$ be irreducible.
If u is in ℓ^2, B is a local perturbation of A, and

$$(A + B)u = \lambda u\,,$$

then u has compact support.

P. Kuchment and B. Vainberg, On the Structure of Eigenfunctions Corresponding to Embedded Eigenvalues of Locally Perturbed Periodic Graph Operators, Commun. Math. Phys. 268 (2006)

A class of operators with reducible Fermi surface

A and L : periodic self-adjoint operators on an n-periodic graph.

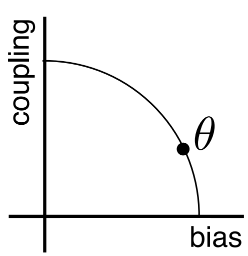
$$\begin{array}{rcl} \text{bias} & : & B = \cos(\theta)\, L \\ \text{coupling} & : & \Gamma = e^{i\phi}\sin(\theta)\, L \end{array} \quad \Longrightarrow \quad L^2 = B^2 + \Gamma\Gamma^*$$

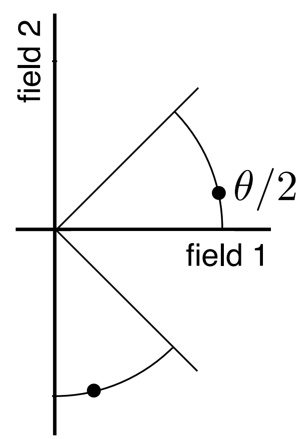
$$\mathcal{A} = \begin{bmatrix} A+B & \Gamma \\ \Gamma^* & A-B \end{bmatrix}, \quad \tilde{\mathcal{A}} = \begin{bmatrix} A+L & 0 \\ 0 & A-L \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} e^{i\phi}\cos(\frac{\theta}{2})I & -\sin(\frac{\theta}{2})I \\ \sin(\frac{\theta}{2})I & e^{-i\phi}\cos(\frac{\theta}{2})I \end{bmatrix}$$

$\Longrightarrow$ $\mathcal{A}\mathcal{U} = \mathcal{U}\tilde{\mathcal{A}}$ with $\mathcal{A}$ and $\tilde{\mathcal{A}}$ self-adjoint and $\mathcal{U}$ unitary.

$$\det\left(\hat{\mathcal{A}}(z) - \lambda I\right) = \det\left(\hat{A}(z) + \hat{L}(z) - \lambda I\right)\det\left(\hat{A}(z) - \hat{L}(z) - \lambda I\right)$$

$\Longrightarrow$ $\Psi_{\mathcal{A},\lambda}$ is reducible for all λ.

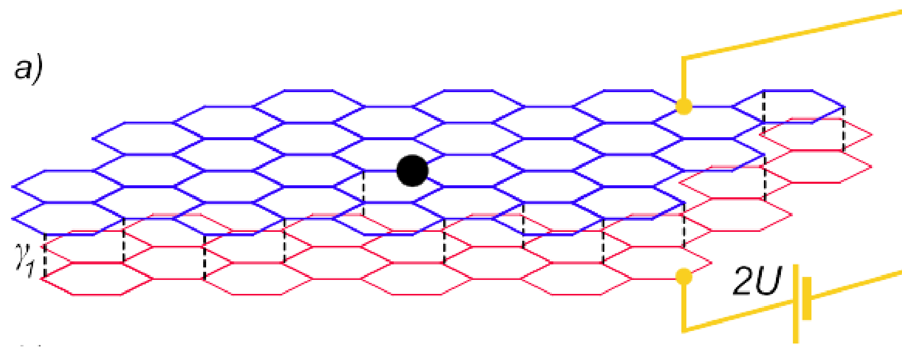
Interpretation: 

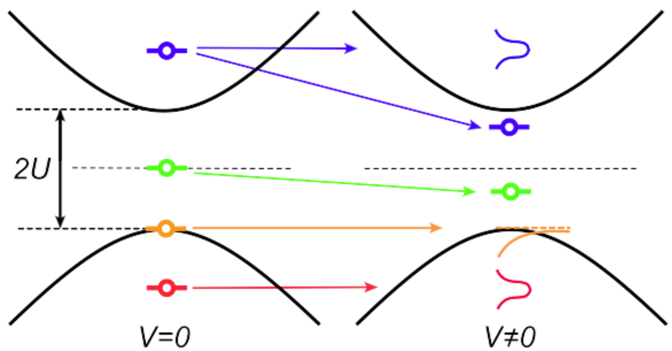
$\Longrightarrow$ 

field components of the columns of $\mathcal{U}$

$\theta = \pi/2 \rightsquigarrow$ even and odd motions

Example: Bilayer graphene





V. V. Mkhitaryan and E. G. Mishchenko, Localized states due to expulsion of resonant impurity levels from the continuum in bilayer graphene, Phys. Rev. Lett. 110, 086805 (2013)

Forcing an evanescent motion within the spectrum

$\tilde{\mathcal{A}}u = f$

decompose u and f into decoupled components: $u = u_+ + u_-$ $f = f_+ + f_-$

$\big(A \pm L - \lambda I\big)u_\pm = f_\pm$

Choose λ such that det $\big(\widehat{A}(z) + \widehat{L}(z) - \lambda I\big)$ has no zeros on the torus $\mathbb{T}^n$
and det $\big(\widehat{A}(z) - \widehat{L}(z) - \lambda I\big)$ does have zeros on $\mathbb{T}^n$

So $\lambda \in \sigma(\mathcal{A})$ but $\lambda \notin \sigma(A + L - \lambda I)$.

$\widehat{u}_+(z) = \big(\widehat{A}(z) + \widehat{L}(z) - \lambda I\big)^{-1}\widehat{f}_+(z)$

$\widehat{u}_-(z) = 0$

Evanescent spectrally embedded eigenfunctions (not compactly supported)

Localized perturbation of A: $A+V$

To construct an embedded eigenvalue for the perturbed operator

$$\mathcal{A}+\mathcal{V}=\left[\begin{array}{cc} A+V+B & \Gamma \\ \Gamma^* & A+V-B \end{array}\right],$$

construct a non-embedded one for $A+V+L-\lambda I$
that is in the spectrum of $A+V-L-\lambda I$.

Want $\big(\widehat{A}(z)+\widehat{L}(z)-\lambda I\big)\widehat{u}_+(z) = -\widehat{V}(z)\widehat{u}_+(z) =: \widehat{f}_+(z)$

Question: Given f_+ and the solution u_+,
can one find a multiplication operator $(Vu)_n = v_n u_{+n}$ such that $v_n u_n = f_{+n}$?

One needs $u_{+n} \neq 0$ for all n such that $f_{+n} \neq 0$.

This is not always possible—but it is if the forcing is localized at one point: $f_{+n} = \delta_{0n}$.