Connections in Mathematical Analysis: The Case of Fourier Series

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INTRODUCTION. Napoleon Bonaparte's expedition to Egypt took place in the summer of 1798, the expeditionary forces arriving on July 1 and capturing Alexandria the following day. On the previous March 27 a young professor at the newly founded *École Polytechnique*, Jean-Joseph Fourier (1768–1830), was summoned by the Minister of the Interior in no uncertain terms [**16**, p. 64]:

> Citizen, the Executive Directory having in the present circumstances a particular need of your talents and of your zeal has just disposed of you for the sake of public service. You should prepare yourself and be ready to depart at the first order.

It was in this manner, perhaps not entirely reconcilable with the idea of *Liberté*, that Fourier joined the Commission of Arts and Sciences of Bonaparte's expedition. The military forces conquered Cairo on July 24, and by August 20 Bonaparte had decreed the foundation of the *Institut d'Egypte* in Cairo to promote the advancement of science in Egypt. Its first meeting, with Fourier appointed as its permanent secretary, was held on August 25.

After several military encounters the French surrendered to invading British forces on August 30, 1801, and were forced to depart from Egypt. Upon his return to France, Fourier resumed his post at the *École Polytechnique* but only briefly. In February of 1802 Bonaparte appointed him *Préfet* of the Department of Isère in the French Alps. It was here, in the city of Grenoble, that Fourier returned to his research endeavors, with which we shall presently occupy ourselves.

But Fourier's stay in Egypt had left a permanent mark on his health that was to influence the direction of his research. He contracted rheumatic pains during the siege of Alexandria and the sudden change of climate, from that of Egypt to that of the Alps, was distressing to him. The facts are that he lived in overheated rooms, that he covered himself with an excessive amount of clothing even in the heat of summer, and that his preoccupation with heat extended to the subject of heat propagation in solid bodies, heat loss by radiation and heat conservation. It was then on the subject of heat that he concentrated his main research efforts.

The results were first presented to the *Institut de France* on December 21, 1807 as a *Mémoire sur la propagation de la chaleur*. It was not entirely well received, and the committee that was to judge it and publish a report on it never did so (it appeared first in [**11**]). Instead, criticisms were made personally to Fourier in one of his visits to Paris in 1808 or 1809. They came mainly from Laplace and Lagrange and referred to two major points: Fourier's derivation of the equations of heat propagation and his use of some series of trigonometric functions known today as *Fourier series*. He replied to these objections and, as a means to settle the question, suggested that a public competition be set up and a prize awarded by the *Institut* to the best work on the propagation of heat. Laplace—who had by then become supportive of Fourier's work—was probably instrumental in converting this suggestion into reality, and this was indeed the subject chosen for a prize essay for the year 1811. Another committee, including Lagrange and Laplace, was to judge on the only two entries, and on January 6, 1812, the prize was awarded to Fourier's *Théorie du mouvement de la chaleur dans les corps solides*. However, the committee's report expressed some reservations, specifically stating that [**11**, p. 452]

> the manner in which the Author arrives at his equations is not exempt from difficulties, and that his analysis, to integrate them, still leaves something to be desired in the realms of both generality and even rigor.

Fourier protested but to no avail, and his new work, like his previous memoir, was not published by the *Institut* at that time. He was to ultimately prevail, and in 1822 he gathered the larger part of his researches on heat in his monumental work *Théorie analytique de la chaleur* [**10**].

There is no doubt that today this book stands as one of the most daring, innovative, and influential works of the nineteenth century on mathematical physics. The methods that Fourier used to deal with heat problems were those of a true pioneer because he had to work with concepts that were not yet properly formulated. He worked with discontinuous functions when others dealt with continuous ones, used integral as an area when integral as an antiderivative was popular, and talked about the convergence of a series of functions before there was a definition of convergence. At the end of his 1811 prize essay, he even integrated 'functions' that have value ∞ at one point and are zero elsewhere. But such methods were to prove fruitful in other disciplines such as electromagnetism, acoustics and hydrodynamics. It was the success of Fourier's work in applications that made necessary a redefinition of the concept of function, the introduction of a definition of convergence, a reexamination of the concept of integral, and the ideas of uniform continuity and uniform convergence. It also provided motivation for the discovery of the theory of sets, was in the background of ideas leading to measure theory, and contained the germ of the theory of distributions. In the remaining sections we shall examine the steps that led from Fourier's work to the development of each of these pillars of classical analysis.

CONVERGENCE AND UNIFORM CONVERGENCE. One of the first problems studied by Fourier was that of a thin bar made of some conducting material, which, for convenience, we shall assume to be of length π and located along the x-axis with endpoints at $x = 0$ and $x = \pi$. If the temperature at a point x at time t is denoted by $u(x, t)$, Fourier deduced that it satisfies the equation

$$u_t = ku_{xx}, \tag{1}$$

where k is a positive constant. If its endpoints are maintained at zero temperature for $t \geq 0$ and if its initial temperature distribution is given by a known function f, we must solve (1) subject to the conditions $u(0, t) = u(\pi, t) = 0$ for $t \geq 0$ and $u(x, 0) = f(x)$ for $0 \leq x \leq \pi$. Fourier found that, for any positive integer n and any real constant c_n, the function $c_n e^{-n^2kt} \sin nx$ is a solution of (1) that vanishes at the endpoints. So is the sum of any number of such functions, but none of these sums need satisfy the initial condition because f may not be a sum of sine functions. Fourier then proposed an infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 kt} \sin nx, \tag{2}$$

and set out to find the constants c_n such that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx = f(x). \tag{3}$$

This is easy if we assume that the last equality holds, if each term of (3) is multiplied by $\sin mx$, and if the resulting expression can be integrated term by term. Then

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx. \tag{4}$$

The series in (3) is a particular instance of a more general form that contains cosine terms in addition to sine terms, the usual *Fourier series*.

Now, the idea that an infinite sum of trigonometric functions can add up to an arbitrary function was rejected by the mathematical establishment. The main obstacle was precisely the concept of function popular at the time. Mathematicians were used to functions given by analytic expressions such as roots, logarithms and so on. How, they demanded, can $f(x) = e^x$ be the sum of an infinite series of sines on an interval $[-\pi, \pi]$? Why, this function is not even periodic while the sine functions are and, consequently, so is the sum of a series of sines. Surprisingly, they failed to realize that it could coincide with a periodic function over a bounded interval. Fourier gave numerous examples in which adding more and more terms of (3), where the c_n are computed from a given function f, results in a sum that is closer and closer to f. But an abundance of examples is not a proof that (3) converges. The problem that mathematicians faced in the early nineteenth century is that there was no definition of convergence. Surely, the concept did exist in some vague manner, but mathematics deals with quantities and comparisons between quantities, with equalities and inequalities. What was needed was a definition of convergence involving comparisons between the partial sums of a series and its proposed sum, such comparisons to be established by means of inequalities. One of the first definitions of convergence along these lines was given by Fourier himself in his prize essay of 1811, later incorporated into his book of 1822. He stated that to establish the convergence of a series [**10**, pp. 196–197]

> it is necessary that the values at which we arrive on increasing continually the number of terms, should approach more and more a fixed limit, and should differ from it only by a quantity which becomes less than any given magnitude: this limit is the value of the series.

The use of inequalities is already implicit in his *less than any given magnitude*. More precise and influential was the definition of convergence given by Augustin-Louis Cauchy (1789–1857). He was the first to understand the importance of rigor in analysis and the first to use inequalities in his definitions of limits and continuity. We shall never know whether or not Fourier's earlier definition helped him in shaping his own ideas. But once in possession of a rigorous definition of limit, Cauchy published the following in his 1821 textbook *Cours d'analyse de l'École Royale Polytechnique* [6, series 2; **3**, p. 114]:

Let $s_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}$ be the sum of the first n terms [of the series under consideration], n being any natural number. If, for always increasing values of n, the sum s_n approaches a certain limit s, the series will be called convergent and the limit in question will be called the sum of the series.

This is essentially the modern definition. More remarkably, Cauchy did not limit himself to stating it. On the next page he gave theorems containing tests for convergence: the Cauchy criterion and the root and ratio tests. A proof of the convergence of Fourier series was attempted by Poisson in 1820, by Cauchy in 1823 and, of course, by Fourier himself throughout his life. He never succeeded, but one of his sketches for a proof [**10**, pp. 438–440] would be of value to the man who finally did.

In 1822 a West Prussian teenager, Johann Peter Gustav Lejeune-Dirichlet (1805–1859), came to Paris to study mathematics. There he became acquainted with Fourier, who encouraged him to complete his sketch of the convergence proof. It would be some time, however, before Dirichlet could do so. In 1829, already a professor at Berlin, he published a paper entitled *Sur la convergence des séries trigonométriques qui servent a représenter une fonction arbitraire entre des limites données* [7, **1**, pp. 117–132]. After replacing a certain trigonometric identity in Fourier's sketch of proof with one of his own, he succeeded in giving sufficient conditions for convergence: if f is piecewise continuous and has a finite number of maxima and minima, then its Fourier series converges to the average of the right-hand and left-hand limits of f at each x.

Dirichlet's theorem is in flagrant contradiction with an earlier one by Cauchy. In his *Cours d'analyse* Cauchy had stated that the sum of a convergent series of continuous functions is continuous [6, series 2, **3**, p. 120]. Already in 1826 Abel had remarked that this theorem is wrong [1, **1**, pp. 224–225], and then, in 1829, Dirichlet's theorem made this abundantly clear. This is not mentioned to show a blemish in Cauchy's work, but because of its connection with an important discovery. Probably at Dirichlet's prompting, one of his students, Phillip Ludwig von Seidel (1821–1896), was led to investigate this matter in 1847. Here is his report: if $\Sigma_{n=1}^{\infty} u_n(x)$ is a convergent series of continuous functions with sum $f(x)$, I is an interval in the domain of these functions, and $\varepsilon > 0$ is given, let N be the smallest positive integer such that

$$\left| \sum_{n=N+1}^{\infty} u_n(x) \right| < \varepsilon$$

for all x in I. Then the given series is said to converge *arbitrarily slowly* on I if $N \to \infty$ as $\varepsilon \to 0$. Using this new concept, that was unavailable to Cauchy in 1821, Seidel was able to prove Cauchy's theorem provided that the convergence is not arbitrarily slow on any interval [**20**]. However, he did not pursue the matter, nor did he realize that he had put forth a powerful new kind of convergence.

As it happens, this idea of a different kind of convergence was not entirely new. Already in 1838 Christof Gudermann (1798–1852) had referred to a kind of convergence at the same rate—*im ganzen gleichen Grad*—that is the precursor of the modern concept of uniform convergence [**13**, pp. 251–252]. But its importance escaped him, as it would escape Seidel later on. This realization was left to Gudermann's student Karl Theodor Wilhelm Weierstrass (1815–1897), one of the giants of modern mathematics. Uninspired by the lectures at the University of Bonn, where he was a student, he went to Münster in 1839 to attend Gudermann's lectures. Gudermann was to influence Weierstrass' research and it is quite likely that, while at Münster, they discussed the new concept of convergence. Weierstrass never finished his doctorate and became a *Gymnasium* teacher in 1841. During his tenure, until 1854, he produced an incredible amount of first-rate research in manuscript form that, regretably, remained unpublished. The fact that he referred to uniform convergence—*gleichmässige Convergenz*—in an 1841 manuscript [23, **1**, pp. 68–69] supports the idea that he may have learned about it from Gudermann. Weierstrass' many research achievements eventually earned him a position at the University of Berlin in 1856, where he frequently discussed uniform convergence. He defined it formally, for functions of several variables, in [23; **2**, pp. 201–233, Art. 1]. Adapted to the one variable case, his definition was:

> An infinite series $\sum_{\nu=0}^{\infty} u_\nu$ converges uniformly in a subset B of the region of convergence if given an arbitrarily small positive quantity δ a whole number m can be found such that the absolute value of the sum $\sum_{\nu=n}^{\infty} u_\nu$ is smaller than δ for each value of $n \geq m$, and for each value of the variable in B.

Still, the importance of Weierstrass' contribution stems from the fact that he realized the usefulness of uniform convergence and incorporated it in theorems on the integrability and differentiability of series of functions term by term.

G. Lejeune-Dirichlet

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THE CONCEPT OF FUNCTION. A lasting controversy over the concept of function started in 1747 when Jean Le Rond d'Alembert (1717–1783), of Paris, published his researches on the vibrating string [**3**]. If a piece of string, initially located along the x-axis and tied down at its endpoints at $x = 0$ and $x = a$, is displaced and then released, and if its vertical displacement at x at time t is denoted by $u(x, t)$, d'Alembert showed that it satisfies the equation

$$u_{tt} = c^2 u_{xx}, \tag{5}$$

where c is a constant. He also showed that if the initial displacement is given by a known function f, then the displacement of the string at any point x and at any time $t \geq 0$ is given by

$$u(x, t) = \tfrac{1}{2}\left[\tilde{f}(x + ct) + \tilde{f}(x - ct)\right],$$

where $\tilde{f}$ is the odd periodic extension of f to $\mathbb{R}$ of period $2a$. It is quite clear that f has to be twice differentiable for u to satisfy (5). However, this differentiability was rejected by Leonhard Euler (1707–1783) who, in a paper of 1748 written at Berlin, allowed a function with a discontinuous derivative as a better model for a plucked string than a twice differentiable function [8, series 2; **10**, pp. 63–77]. d'Alembert would not accept such functions [**2**], and this disagreement marked the beginning of a lively mathematical argument between the two men. The fact is that Euler's proposal represented something very new, since the concept of function at the time was that of an analytic expression or formula. In fact, this was the year of publication of Euler's enormously influential treatise *Introductio in analysin infinitorum* [8, series 1, **8** and **9**], the standard text on analysis for the next half century. At the very beginning, in the fourth paragraph, he defined a function of a variable quantity as

> any analytic expression made up in any manner whatever from that variable quantity and numbers and constants.

But then, that very same year, the vibrating string problem made him realize that this definition was too narrow to fit the needs of applied mathematics.

d'Alembert's solution completely describes the motion of the string, for it specifies the position of each of its points at each time. Mathematically that is all very well, but where is the musical description of the phenomenon? Where are the vibrations? This solution does not show a periodicity in t. It was Euler who stated that the motion of the string is periodic in time and made up of individual vibrations. In fact, in 1748 he wrote down the equation

$$u(x, t) = \sum c_n \sin \frac{n\pi}{a} x \cos \frac{n\pi}{c} t, \tag{6}$$

meant to be valid only if f is a sum of sines, but did not specify whether these sums are finite or infinite. Upon reading d'Alembert's and Euler's papers, Daniel Bernoulli (1700–1782), of Basel, decided to publish his own ideas on the subject, which he did in 1753 [**4**]. Perhaps there was an element of irritation in the fact that Euler now stated what he had known for some time. In a previous paper Bernoulli had already stated that the shape of the string at a given instant is the superposition of individual vibrations. Now, after having a bit of fun criticizing d'Alembert and Euler—he referred to the former as a great mathematician *in abstractis*—he asserted that this shape can be represented by an infinite series of sines. In particular, for $t = 0$,

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{a} x. \tag{7}$$

If we accept this equation, we can combine it with (6) to arrive at the following expression for the solution of the vibrating string problem.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{a} x \cos \frac{n\pi}{c} t.$$

Although Bernoulli never actually wrote this equation, it is called nowadays *Bernoulli's solution*, and it clearly shows that the motion of the string is periodic in time. Bernoulli based his equation (7) on physical considerations alone and provided no mathematical reasons whatsoever to back it up. Euler pounced on it immediately, the very same year, refusing to accept it [8, series 2; **10**, pp. 232–254]. For one thing, its right-hand side is a periodic function, which f need not be. Moreover, harping on his earlier idea that f need not be differentiable at all points, he rejected (7) because the sine functions on the right are differentiable. d'Alembert published a similar attack on Bernoulli's paper, but he did not surrender his position for, he said, he had infinitely many coefficients to choose to make the equality true. All this created a heated controversy that raged through the 1770's, without any of the participants giving an inch to the others' point of view. It was later revived through Fourier's researches on heat and eventually settled once and for all: the sum of an infinite series of sines can be a function that is not differentiable at all points.

With all this, Euler's wider concept of function emerged as the winner over the idea of function as a formula. In his *Institutiones calculi differentialis* of 1755, Euler himself gave the new definition as follows [8, series 1; **10**, p. 4]:

> If some quantities depend on other quantities so that they change when the latter are varied, then the former quantities are called functions of the latter.

This would not be the last word, however. For one thing, it is vague, lacking the precision demanded by the publication of Cauchy's *Cours d'analyse*. For another it was not totally accepted. What definitely won the day was Fourier's work, his use of discontinuous functions, and Dirichlet's proof of Fourier's assertion that a trigonometric series could converge to such a function. After this there was no turning back to the purely analytic concept of function. Fourier himself tried his hand at a new definition as follows [**10**, p. 432]:

> The function $f(x)$ denotes a function completely arbitrary, that is to say a succession of given values, subject or not to a common law, and answering to all the values of x between 0 and any magnitude X.

But, in spite of this *completely arbitrary* qualifier (what does it mean, anyway?), it is clear from an examination of his work that Fourier never had in mind a function with more than a finite number of discontinuities.

Neither did Dirichlet up to a point. But then he realized that a full generalization of his convergence theorem should allow integrable functions with infinitely many discontinuities [7, p. 131]. If this motivated him to search for a general definition of function, then he must have lost track of what he was after for the fact is that, contrary to what many have asserted, he never stated such a definition. Later on, during the years 1847–1849, Dirichlet had the good fortune of counting a very gifted young man among his students at the University of Berlin. Georg Friedrich Bernhard Riemann (1826–1866) had transferred from the University of Göttingen to Berlin, and here Dirichlet was his favorite teacher and was instrumental in shaping some of Riemann's research interests. We do not know whether or not they discussed the concept of function before Riemann returned to Göttingen, where he received his doctorate in 1851. The fact is that in the opening paragraphs of his thesis we read [**18**, p. 3]:

> If we let z be a variable quantity that can gradually assume all possible real values, when to each of its values there corresponds a unique value of the undetermined quantity w, then we say that w is a function of $z \ldots$ this definition does not specify any fixed law between the individual values of the function, because, after it is defined on a particular interval, the way it can be extended outside remains entirely arbitrary.

Bernhard Riemann

Dirk J. Struik, *A Concise History of Mathematics*, 1948, Dover Publications, Inc., New York. Reprinted with permission.

Which is what Fourier had been saying all along: no *common law*, and it does not matter how the function is extended beyond $[-\pi, \pi]$. But with Riemann we have precision, we have this correspondence of a unique value of the function to each value of the variable. In short, the first entirely general and modern definition of function. With it ends, once and for all, an era of misconception. For it may once have been believed, when functions were just given by analytic expressions, that every continuous function has a derivative but not necessarily an integral. In fact, the opposite is true: not every continuous function has a derivative, while they all have integrals. But this is another topic.

INTEGRATION. The popular concept of integral in the eighteenth century was that of antiderivative. Leibniz had defined the integral much earlier as a sum, but his idea did not quite catch for some time. How could it, involving, as it does, the sum of infinitely many infinitely small quantities? Fourier changed that. He was used to handling functions not given by analytic expressions, but by curves and pieces of curves, and found antiderivatives to be impractical. Instead he remarked that, whether or not f is continuous, the integral defining the constant c_n in (4) can be viewed as the area under the graph of $f(x)\sin nx$ from 0 to π [**10**, p. 186]. It may have been responding to this interpretation of the integral as an area that Cauchy gave the following definition in his *Resumé des leçons donnés à l'École Royale Polytechnique sur le calcul infinitésimal* of 1823 [6, series 2, **4**, p. 125], which we reproduce in the current notation. If f is continuous on an interval $[a, b]$ and if $x_0, x_1, \dots, x_n$ are points such that $a = x_0 < x_1 < \cdots < x_n = b$, then

$$\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}), \tag{8}$$

provided that $x_i - x_{i-1} \to 0$ for each i as $n \to \infty$. Cauchy was then able to prove —not rigorously because he lacked the concept of uniform continuity—the existence of this limit. Notice also that if f is piecewise continuous it is still integrable because $[a, b]$ can be partitioned into a finite number of subintervals where f is continuous, and then the integrals over each of these subintervals can be added together. Incidentally, this notation for the definite integral, adopted by Cauchy, is due to Fourier [**10**, p. 463].

This definition suffices to prove Dirichlet's convergence theorem. In fact, Dirichlet had limited the discontinuities of his functions to a finite number to make them integrable. In order to generalize the theorem to functions with infinitely many discontinuities, he only needed to make sure that they could be integrated. That is, what he needed is what Cauchy's definition did not provide, namely, a condition for integrability. Dirichlet never achieved his goal of integrating functions with infinitely many discontinuities, but Riemann, who had acquired an interest in these topics from Dirichlet, would succeed. In 1854, wishing to qualify for a position at Göttingen as *Privatdozent*, he wrote a *Habilitationsschrift*, which at Dirichlet's suggestion was *Über die Darstellbarkeit einer Function durch eine trigonometrische Reiche*. Here he modified Cauchy's definition by replacing the factor $f(x_{i-1})$ in (8) by $f(t_i)$, where t_i is any point in the subinterval $[x_{i-1}, x_i]$, and by removing the continuity requirement on f. Instead, he turned things around and *defined* f to be integrable if the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \tag{9}$$

exists, provided that, for each $i, x_i - x_{i-1} \to 0$ as $n \to \infty$ [**18**, p. 239]. Next he stated a theorem giving conditions for the integral to exist [**18**, pp. 240–241], and to show the wide applicability of his definition, he gave an example of an integrable function with infinitely many discontinuities [**18**, p. 242].

Of course, not every function is integrable. For instance, at the end of his 1829 paper, Dirichlet pointed out that if c and d are constants and if $f(x) = c$ when x is rational and $f(x) = d$ when x is irrational, then the integrals that define the Fourier coefficients of f lose all significance [**7**, p. 132]. Indeed, the sum in (9) has value c if each t_i is rational and value d if each t_i is irrational, so that the limit does not exist. However, this is a rather weird function and the fact that it is not integrable was regarded as unimportant. It seemed for quite some time that Riemann's definition of integral was the most general imaginable. Reality, in its usual fashion, would soon dispel this illusion.

THE THEORY OF SETS. The coefficients in (3) were obtained by assuming that the series converges and can be integrated term by term. Can it? A theorem of Weierstrass states that it can if the convergence is uniform. Then we ask: when does a Fourier series converge uniformly? We are not just posing a purely theoretical question because the needs of applications demand an answer. For instance, in order for (2) to be the solution of the problem posed earlier, it must be continuous for $t \geq 0$ and $0 \leq x \leq \pi$. This is true if (2) converges uniformly, as shown by Abel, unknowingly using the idea of uniform convergence [1; **1**, pp. 224–225]. But then, in particular, the convergence of (2) must be uniform for $t = 0$, that is, the Fourier series in (3) must be uniformly convergent. So, once again, when does a Fourier series converge uniformly? This is the question that Heinrich Eduard Heine (1821–1881), of the University of Halle, posed himself, and in 1870 he showed that if a function satisfies Dirichlet's conditions on $[-\pi, \pi]$, then its Fourier series converges uniformly on the set that results after removing arbitrarily small neighborhoods of the points where it is discontinuous [**15**].

Now, in his integration paper Riemann had also considered trigonometric series on $[-\pi, \pi]$ of the usual form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{10}$$

but with arbitrary coefficients, not necessarily the Fourier coefficients of some function [**18**, p. 245]. In principle, there may be several choices of the coefficients for which (10) converges to the same function. But this is impossible if (10) converges uniformly, for then term by term integration shows that they must be the Fourier coefficients of its sum. It was at this point that Heine posed a second problem: how to weaken the hypothesis of uniform convergence and still be able to conclude that the coefficients are unique. He found that if (10) converges uniformly on the subset of $[-\pi, \pi]$ that remains after removing arbitrarily small neighborhoods of a finite number of points, then the coefficients are unique [**15**].

Notice that Heine, even though geographically removed from the Weierstrassian world at Berlin, used uniform convergence. He had been a student of Weierstrass and may have learned about it before leaving Berlin, or he may have heard about it from a new arrival from Berlin, Georg Ferdinand Louis Philippe Cantor (1845–1918), who had become a *Privatdozent* in 1869 at Halle. In any case, Heine encouraged Cantor to do some further work on the problem of uniqueness of the coefficients of (10). Cantor started with the idea of discarding uniform convergence entirely, and succeeded fairly soon, but had to assume that (10) converges at every point [**5**, pp. 80–83]. Then, in 1871, he was able to allow (10) to diverge a finite number of points and still conclude that its coefficients are unique [**5**, pp. 84–86]. But Cantor was ambitious and found these results short of what he wanted to do, namely to reach the same conclusion after allowing the convergence of (10) to fail at infinitely many points. But then, what kind of infinite set of points should this be? In 1872, Cantor found that, in order to construct such a set, he needed to develop first a theory of the real numbers. Having accomplished this, he defined the concept of limit point [**5**, p. 98]:

> Given a set of points P, if there are an infinite number of points of P in every neighborhood, no matter how small, of a point p, then p is said to be a limit point of the set P.

By a neighborhood of p Cantor meant an open interval containing p. Then he defined the *derived set* P' of P as the set of all limit points of P, the second derived set P'' of P as the derived set of P', and so on until, after k iterations, the k-th derived set $P^{(k)}$ of P is the derived set of $P^{(k-1)}$. Then he proved his most general uniqueness theorem in the following form: if (10) vanishes for all values of x in $[-\pi, \pi]$ except for those corresponding to a subset P such that $P^{(k)}$ is empty for some k, then all its coefficients are zero [**5**, p. 99].

Having found his motivation on questions about trigonometric series, Cantor had just laid the foundations on which he would then build his acclaimed and controversial theory of sets.

MEASURE-THEORETIC INTEGRATION. This is, then, the way it was: in 1870 Cantor gave the first steps toward the theory of sets by investigating the set of points where (10) may fail to vanish and still conclude that $a_n = b_n = 0$. This is, instead, the way it could have been: in 1870 Hermann Hankel (1839–1873) could have given the first steps toward the theory of sets by investigating the set of points where a function may be discontinuous and still integrable. A professor at Tübingen, Hankel had been a student of Riemann at Göttingen and was seeking a necessary and sufficient condition for integrability. In view of Riemann's example of a highly discontinuous integrable function, Hankel wanted to characterize integrability in terms of the set of points where a function is discontinuous, and started by defining the *jump* of f at a point x_0 to be the largest—*i.e.*, the supremum—of all numbers $\sigma > 0$ such that in any interval containing x_0 there is an x for which $|f(x) - f(x_0)| > \sigma$ [**14**, p. 87]. Then, if S_σ denotes the set of points where the jump of f is greater than σ, Hankel concluded that a bounded function is integrable if and only if for every $\sigma > 0$ the set S_σ can be enclosed in a finite collection of intervals of arbitrarily small total length, a fact that we express by saying that *S_σ has content zero*. On the other hand, if a set cannot be so enclosed it is said to have *positive content*. With this result Hankel initiated the set-theoretic approach to integration.

But instead of developing these ideas, Hankel next made a mistake and stated the wrong theorem. First he defined a set to be *scattered*—the modern term, due to Cantor, is *nowhere dense*—if between any two of its points there is an entire interval that contains no points of the set. And then, erroneously thinking that a set has content zero if and only if it is scattered, he stated that a bounded function is integrable if and only if for every $\sigma > 0$ the set S_σ is scattered. Henry John Stephen Smith (1826–1883), of Oxford, carefully read Hankel's paper, found the error and, in 1875, gave several methods to construct nowhere dense sets of positive content [**21**, p. 148]. It is easy to see that if S is one such set contained in an interval I and if $f \equiv 1$ on S and $f \equiv 0$ on $I - S$ then f is not integrable.

Then, in 1881 Vito Volterra (1860–1940), a student at Pisa, used a nowhere dense set of positive content to construct a function f on $[0, 1]$ such that f' exists and is bounded at every point, but is not integrable [**22**]. Therefore, while f' always has an integral in the sense of antiderivative, it may not have an integral in Riemann's sense. It can then be said that Riemann's definition is beginning to show some rough edges. Furthermore, it was known, at least since 1875, that it is not always possible to interchange passage to the limit and integration in a sequence of integrable functions.

All this meant that the definition of integrability had to come up for review and, in view of Hankel's characterization of it in terms of sets of content zero, the new approach had to be set-theoretic. After some preliminary work by Marie-Ennemond Camille Jordan (1838–1922), this was accomplished by Henri-Léon Lebesgue (1875–1941) in his doctoral dissertation of 1902 at the Sorbonne, later expanded into a book [**17**]. Here he introduced a theory of the measure of sets and, based on it, a definition of integral that generalizes that of Riemann but is free of the defects pointed out above [**17**, pp. 110–121].

THE THEORY OF DISTRIBUTIONS. In his 1811 memoir Fourier considered heat propagation in an ideal bar of infinite length whose initial temperature is a known function f. A series solution was not possible in this case and he proposed, instead, an integral solution. To satisfy the initial condition, it must equal f for $t = 0$, leading—in modern notation—to the integral equation

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega, \tag{11}$$

that must be solved for the unknown function $\hat{f}$. The solution is

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx. \tag{12}$$

Fourier's proof was unrigorous but interesting because it contains the germ of further discoveries, and we shall examine it next. If we substitute (12) into the right-hand side of (11), reverse the order of integration and simplify, we obtain

$$\int_{-\infty}^{\infty} f(s) \left(\frac{1}{\pi} \int_0^{\infty} \cos \omega(x-s)\, d\omega \right) ds = \int_{-\infty}^{\infty} f(s) \frac{1}{\pi} \lim_{p \to \infty} \frac{\sin p(x-s)}{x-s}\, ds. \tag{13}$$

Then Fourier stated that the right-hand side is equal to

$$\int_{-\infty}^{\infty} f(s) \frac{\sin p(x-s)}{\pi(x-s)}\, ds, \tag{14}$$

where, he said, $p = \infty$. Let's just say that if $p > 0$ is fixed and very large (14) is an approximation of the right-hand side of (11). For p very large, $\sin p(x-s)$ undergoes a complete oscillation on every interval $[x + k\pi/p, x + (k+2)\pi/p]$, where k is any integer, and $f(s)/(x-s)$ is approximately constant in each for $k \neq -1$. In the remaining interval $f(s) \approx f(x)$, and then

$$\int_{-\infty}^{\infty} f(s) \frac{\sin p(x-s)}{\pi(x-s)}\, ds \approx f(x) \int_{x-\pi/p}^{x+\pi/p} \frac{\sin p(x-s)}{\pi(x-s)}\, ds = f(x) \int_{-\pi/p}^{\pi/p} \frac{\sin pu}{\pi u}\, du.$$

But, as above, the integral of the quotient on the right over the rest of the real line is negligible, and then

$$\int_{-\infty}^{\infty} f(s) \frac{\sin p(x-s)}{\pi(x-s)}\, ds \approx f(x) \int_{-\infty}^{\infty} \frac{\sin pu}{\pi u}\, du = \frac{f(x)}{\pi} \int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = f(x). \tag{15}$$

Fourier, however, kept $p = \infty$ throughout his argument [**10**, pp. 426–429]. It seems that he would have us believe that there is a function δ defined by

$$\delta(x) = \lim_{p \to \infty} \frac{\sin px}{\pi x}$$

and such that, as suggested by (15),

$$\int_{-\infty}^{\infty} f(s)\delta(x-s)\,ds = f(x). \tag{16}$$

(15) also suggests that the integral of δ over the whole real line is one, while the argument following (14) shows that its integral over any interval that excludes the origin is zero. In short, $\delta \equiv 0$ outside the origin and $\delta(0) = \infty$.

There is, of course, no such function. But we wish there were for the sake of applications. For instance, in *An essay on the application of mathematical analysis to the theories of electricity and magnetism* of 1828, George Green (1793–1841), of Cambridge, considered the problem of solving the equation

$$u_{xx} + u_{yy} + u_{zz} = f \tag{17}$$

in a bounded region of space that contains the origin. Here u is the electrostatic potential created by a charge distribution given by f. He showed that he could solve this problem if he could first solve it for the restricted case in which there is just one point charge—infinite charge density—at the origin and none elsewhere [**12**, pp. 32–33]. Now, let's say that there is a δ function on $\mathbb{R}^3$ with the properties listed above except that the integrals are three-dimensional. Since, in particular, $\delta \equiv 0$ outside the origin and $\delta(0) = \infty$, we can rephrase Green's claim as follows: a solution of (17) can be obtained from a solution of

$$u_{xx} + u_{yy} + u_{zz} = \delta. \tag{18}$$

Indeed, let u^{δ} be a solution of (18), denote the function of x defined by the left-hand side of (16) by $f * \delta$, and define $f * u^{\delta}$ in the same way but replacing δ with u^{δ} in the integrand. Then $u = f * u^{\delta}$ is a solution of (17) because, if differentiation under the integral sign is permitted,

$$u_{xx} + u_{yy} + u_{zz} = f * \left(u^{\delta}_{xx} + u^{\delta}_{yy} + u^{\delta}_{zz}\right) = f * \delta = f,$$

where the last equality is just (16).

The power of wishful thinking cannot be underestimated. During the period 1945–1948 Laurent Schwartz (1915–), working in isolation at Grenoble as Fourier had done before, developed a complete, rigorous, and applicable theory of this δ and similar 'functions', which he called *distributions*, culminating in the publication of his two-volume work *Théorie des distributions* [**19**].

EPILOGUE. Back in 1811, disappointed by the committee's reaction to his memoir, Fourier returned to Grenoble and, being far from Paris, lacked the power and the influence to have his prize essay published by the *Institut*. But new political events would soon change his fortune. A European Alliance against Napoleon forced his unconditional abdication on April 11, 1814, restoring the monarchy in the person of Louis XVIII. Fourier remained as *Préfet* of Isère under the new regime, a tribute to his diplomatic abilities, but early the following March he learned that Napoleon had returned from his exile at Elba. Fearing the consequences of his temporary allegiance to the Crown, he fled to Lyons, but by the time he arrived there the Emperor had forgiven his ungrateful behavior and appointed him *Préfet* of the Rhône. He was dismissed from this position on

May 17 and, having been granted a pension of 6,000 francs by Napoleon, Fourier finally returned to Paris. A new allied army defeated Napoleon on June 18, 1815, at the Battle of Waterloo, and he was forever banished to the island of St. Helena. Fourier's pension never materialized under the King's restored government, and he found himself penniless. However, with the influence of a friend and former student at the *École Polytechnique*, the Count of Chabrol de Volvic, he secured the position of Director of the Bureau of Statistics of the Department of the Seine, and this allowed him to remain in Paris permanently and to set down to business.

First, there was the publication of the prize essay, a matter in which he succeeded after a considerable amount of insistence. It finally appeared in 1824 and 1826 in volumes 4 and 5 of the *Mémoires de l'Académie Royale des Sciences de l'Institut de France* [9; **2**, pp. 1–94]. But before this, in May of 1816, two new members of the Academy of Sciences were to be elected. Fourier lobbied vigorously on his own behalf and, after several rounds of voting, was elected to the second position. The King, resentful of Fourier's activities during Napoleon's second period in power, refused to give his approval. But a regular vacancy was created again in 1817, and on the election of May 12 Fourier obtained forty seven of the fifty votes. The King was then compelled to grant his approval.

Fourier's scientific standing was no longer in doubt. In 1822 his *Théorie analytique de la chaleur* was printed in Paris, and on November 18 of the same year he became Permanent Secretary of the mathematics section of the Academy of Sciences. His last years were marked by honors and poor health. He was elected to the Royal Society of London and to the *Académie Française* in 1826. Then, the next year, he became president of the *Conseil de perfectionnement de l'École Polytechnique*. But already in 1826, in a letter to Auger, permanent secretary of the French Academy, he claimed to *see the other bank where one is healed of life* [**16**, p. 137]. In addition to his rheumatism, which never left him, he developed a shortness of breath that was particularly acute if not standing up. Resourceful to the very end, he invented a contraption in the form of a box with holes for his arms and head to protrude, and carried on in this fashion. The end came at about four o'clock in the afternoon of May 16, 1830 in the form of a heart attack, and shortly afterward he died.

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