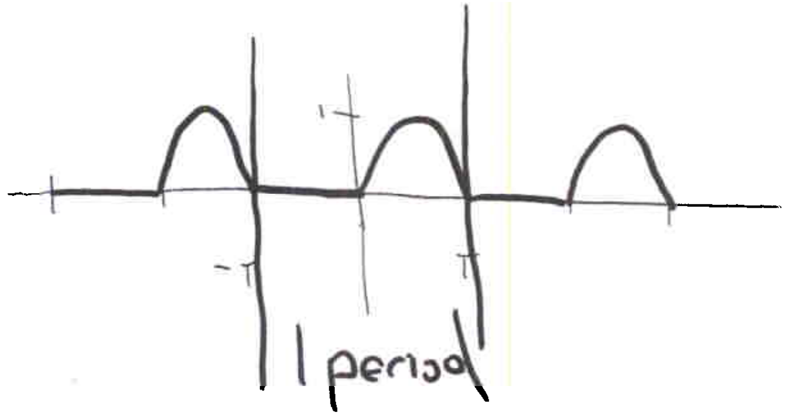
7.5]

11.) $f = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx = \frac{1}{\pi}\left[\frac{2}{1-n^2}\right] \quad n \text{ even}$$
$n \neq 0$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x = \frac{-1}{\pi}\Big|_0^{\pi} \cos x = \frac{-1}{\pi}\left[-1-1\right] = \frac{2}{\pi}$$

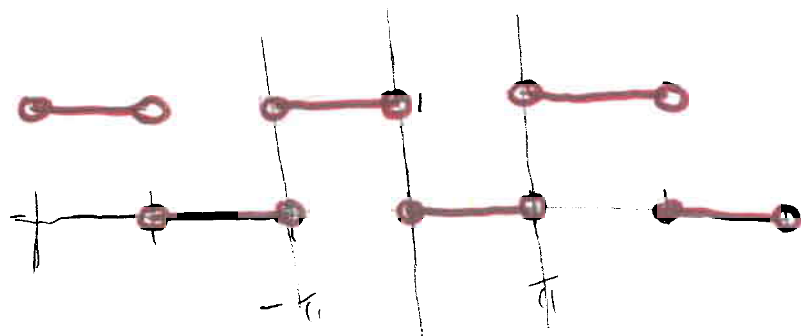
$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\,dx \overset{n=1}{=} \frac{1}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{2}{1-n^2}\right] \quad n \text{ even}, \; n \neq 0, \qquad a_0 = \frac{2}{\pi}$$

$$b_1 = \frac{1}{2}$$

7.6

1.) $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}$



the expansion converges to the exact value where $f(x)$ is continuous

$f\left(\frac{\pi}{2}\right) = 0$

$f\left(-\frac{\pi}{2}\right) = 1$

the expansion converges to the average value of $f(x)$ where $f(x)$ is discontinuous

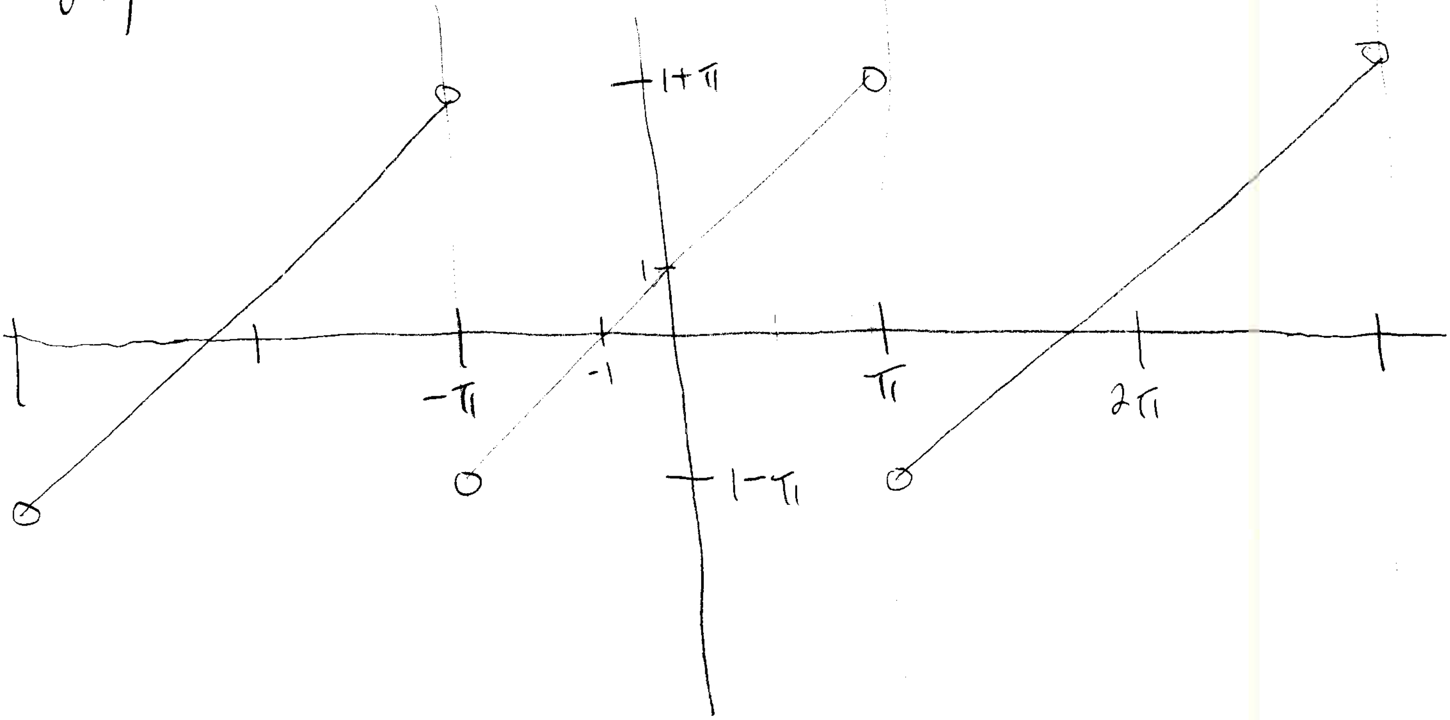
$f(0) = \frac{0+1}{2} = \frac{1}{2}$

$f(\pm\pi) = \frac{1}{2}$

$f(\pm 2\pi) = \frac{1}{2}$

7.6

8.) $f(x) = 1 + x \quad -\pi < x < \pi$



$f(0) = 1$

$f\left(\frac{\pi}{2}\right) = 1 + \frac{\pi}{2}$

$f\left(-\frac{\pi}{2}\right) = 1 - \frac{\pi}{2}$

$f(\pm\pi) = \frac{1 + \cancel{\pi} + 1 - \cancel{\pi}}{2} = 1$

$f(\pm 2\pi) = f(0) = 1$

7.1

1.) $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f e^{-inx}\,dx$ $C_0 = \frac{1}{2\pi}\int_{-\pi}^{0} 1\,dx = \frac{1}{2}$

$n \neq 0$ $= \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}$

$= \frac{-1}{2\pi i n}\left[1 - e^{in\pi}\right]$

$= \frac{-1}{\pi i n}$ n odd

$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} = \frac{1}{2} - \frac{1}{\pi i}\left[\frac{e^{ix}}{1} + \frac{e^{i3x}}{3} + \dots\right]$

$- \frac{1}{\pi i}\left[\frac{e^{-ix}}{-1} + \frac{e^{-i3x}}{-3} + \dots\right]$

use $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$= \frac{1}{2} - \frac{2}{\pi}\left[\frac{e^{ix} - e^{-ix}}{2i} + \frac{1}{3}\left(\frac{e^{i3x} - e^{-i3x}}{2i}\right) + \dots\right]$

$= \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \dots\right]$

7.1

8.) $f(x) = 1 + x \quad -\pi < x < \pi$

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 + \overset{\text{odd fn}}{x}\, dx = \frac{2\pi}{2\pi} = 1 = C_0$$

has period 2π

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1+x)e^{-inx} = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \cancel{e^{-inx}}^{\,0} + \int_{-\pi}^{\pi} x e^{-inx}\right]$$

$$(xe^{-inx})' = e^{-inx} + -inxe^{-inx}$$ have

$$= \frac{1}{2\pi}\cdot\frac{1}{-in}\left[\Big|_{-\pi}^{\pi} xe^{-inx} - \int_{-\pi}^{\pi} \cancel{e^{-inx}}^{\,0}\right]$$

$$= \frac{-1}{2\pi in}\left(\pi e^{-in\pi} - -\pi e^{in\pi}\right) = \frac{-\cancel{\pi}}{2\cancel{\pi} in}\left(e^{-in\pi} + e^{in\pi}\right)$$

$$C_n = \frac{(-1)^{n+1}}{in}$$

$$f(x) = 1 + \frac{1}{i}\left(\frac{e^{ix}}{1} - \frac{e^{i2x}}{2} + \frac{e^{i3x}}{3} - \cdots\right)$$

$$+ \frac{1}{i}\left(-\frac{e^{-ix}}{1} + \frac{e^{-i2x}}{2} - \frac{e^{-i3x}}{3} + \cdots\right)$$

$$= 1 + 2\left(\left(\frac{e^{ix} - e^{-ix}}{2i}\right) - \frac{1}{2}\left(\frac{e^{i2x} - e^{-i2x}}{2i}\right) + \cdots\right)$$

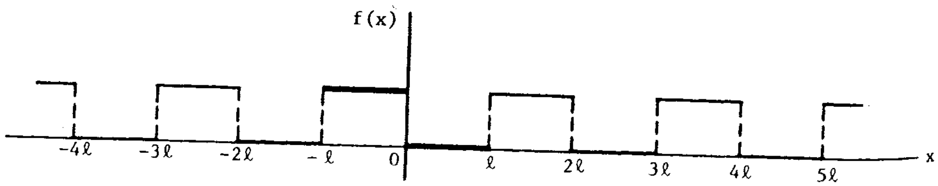
$$= 1 + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

Section 8

Problems 1 to 8, hint: If you put $\ell = \pi$ in your answers, you should get the answers to the corresponding problems given in Section 5.

1. We first sketch several periods of the function to be expanded:

$$f(x) = \begin{cases} 1, & -\ell < x < 0, \\ 0, & 0 < x < \ell. \end{cases}$$



1. (continued)

Now compare the sketch with Figure 8.3 of the text; we see that they are identical. Although the basic interval used to define f(x) is different in the two cases, the periodic functions are the same. Since the average value of a periodic function over a period is the same no matter which period we use, the values of c_n (= average of $f(x)e^{-inx}$) will be the same for this problem and the text example, and so the Fourier series are the same. In the text, c_n was found as an integral from ℓ to 2ℓ. Here we would naturally find c_n as an integral from $-\ell$ to 0. By direct evaluation, we find

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{0} e^{-in\pi x/\ell}\,dx = \frac{1}{2\ell(-in\pi/\ell)}\, e^{-in\pi x/\ell}\Big|_{-\ell}^{0} = \frac{1 - e^{in\pi}}{-2\pi in}$$

as in the text. Similarly the a_n's and b_n's are the same as for the text example.

7.8

8.) $f(x) = 1 + x \quad -\pi < x < \pi$

now on $(-l, l)$

$$c_0 = \frac{1}{2l}\int_{-l}^{l} (1 + \cancel{x}^{\,0})\,dx = \frac{2l}{2l} = 1$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} (1+x) e^{\frac{-in\pi x}{l}} = \frac{1}{2l}\left[\int_{-l}^{l} \cancel{e^{\frac{-in\pi x}{l}}}^{\,0} + \int_{-l}^{l} x e^{\frac{-in\pi x}{l}}\right]$$

$$= \frac{1}{2\cancel{l}} \cdot \frac{-\cancel{l}}{in\pi} \Big|_{-l}^{l} x e^{\frac{-in\pi x}{l}} = \frac{-1}{2in\pi}\left[l e^{-in\pi} + l e^{in\pi} \right]$$

$$= \frac{-l}{\cancel{2}in\pi} \cdot \cancel{2} \cdot (-1)^n = \frac{-l}{in\pi}(-1)^n$$

$$f(x) = 1 - \frac{l}{i\pi}\left[-e^{\frac{i\pi x}{l}} + \frac{e^{\frac{i2\pi x}{l}}}{2} + \cdots \right]$$

$$-\frac{l}{i\pi}\left[+e^{\frac{-i\pi x}{l}} - \frac{e^{\frac{i2\pi x}{l}}}{2} + \cdots \right]$$

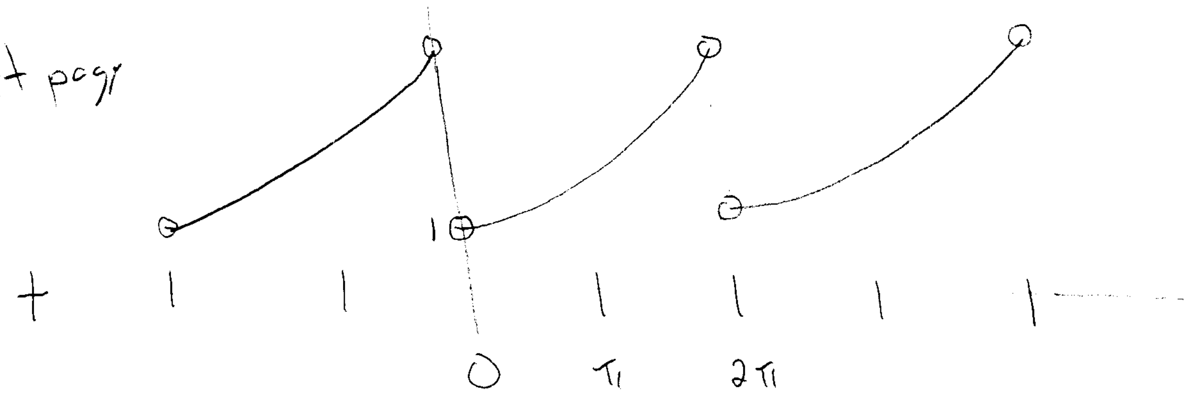
$$f(x) = 1 + \frac{2l}{\pi}\left[\left(\frac{e^{\frac{i\pi x}{l}} - e^{-\frac{i\pi x}{l}}}{2i}\right) - \frac{1}{2}\left(\frac{e^{\frac{i2\pi x}{l}} - e^{-\frac{i2\pi x}{l}}}{2i}\right) + \cdots\right]$$

$$= 1 + \frac{2l}{\pi}\left[\sin\left(\frac{\pi x}{l}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{l}\right) + \cdots\right]$$

7.8

12.a) next page

12.b)



$n \neq 0$

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} e^{x} e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{x(1-in)}\,dx$$

$$= \frac{1}{2\pi(1-in)}\left(e^{2\pi(1-in)}-1\right) = \frac{e^{2\pi}-1}{2\pi(1-in)}$$

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{x} = \frac{1}{2\pi}\left(e^{2\pi}-1\right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{e^{2\pi}-1}{2\pi(1-in)}\right) e^{inx}$$

$$a_0 = \frac{\left(e^{2\pi}-1\right)}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx = \frac{1}{\pi}\Big|_0^{2\pi} \frac{e^x(\cos nx + n\sin nx)}{1+n^2}$$

$$= \frac{1}{\pi}\left(\frac{e^{2\pi}}{1+n^2} - \frac{1}{1+n^2}\right) = \frac{1}{\pi}\left(\frac{e^{2\pi}-1}{1+n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx = \frac{1}{\pi}\Big|_0^{2\pi} \frac{e^x(\sin nx - n\cos nx)}{1+n^2}$$

$$= \frac{1}{\pi}\left(\frac{-e^{2\pi}n + n}{1+n^2}\right) = \frac{-n}{\pi}\left(\frac{e^{2\pi}-1}{1+n^2}\right)$$

$$f(x) = \frac{e^{2\pi}-1}{2\pi} + \frac{e^{2\pi}-1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{1+n^2}\cos nx - \frac{n}{1+n^2}\sin nx\right)$$

12.) a.) to get these expansions just replace x by $z+\pi$ and let $-\pi < z < \pi$

$\overline{14.a)}^{\,7.8}$ $\alpha_x x + \beta e_y y$

14.b)

$$C_0 = \frac{1}{\cancel{2}\frac{1}{\cancel{2}}}\int_0^1 \sin \pi x = \frac{-1}{\pi}\Big|_0^1 \cos \pi x = \frac{-1}{\pi}\Big(-1-1\Big) = \frac{2}{\pi}$$

$$\underset{n\neq 0}{C_n} = 1\int_0^1 \sin \pi x \, e^{-i2n\pi x}\, dx$$

$$= \int_0^1 \sin \pi x \cos(2n\pi x) - i\int_0^1 \cancel{\sin \pi x \sin(2n\pi x)}^{\;0}$$

first $l_y + u = \pi x$

$$= \frac{2}{\pi\left(1-(2n)^2\right)}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi}\left(\frac{1}{1-4n^2}\right) e^{i2\pi n x}$$

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi}\left[\frac{e^{i2\pi x}}{-3} + \frac{e^{i4\pi x}}{-15} + \cdots\right]$$

$$+ \frac{2}{\pi}\left[\frac{e^{-i2\pi x}}{-3} + \frac{e^{-4\pi x}}{-15} + \cdots\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2\pi x}{3} + \frac{\cos 4\pi x}{15} + \cdots\right]$$

14.1 a.) to get your expansion on the interval $-\frac{1}{2} < x < \frac{1}{2}$ you just need to shift

your domain

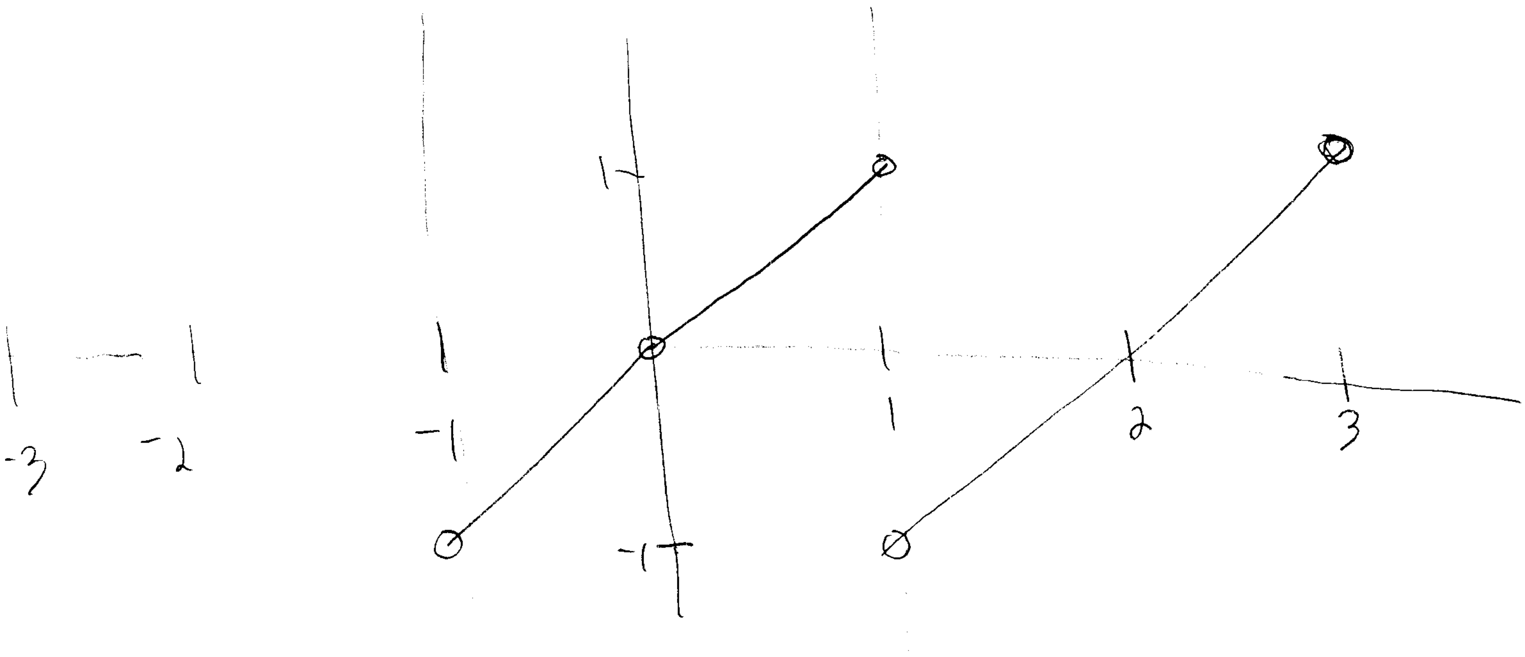
let $-\frac{1}{2} < z < \frac{1}{2}$ and replace your x by $z + \frac{1}{2}$

in your expansion

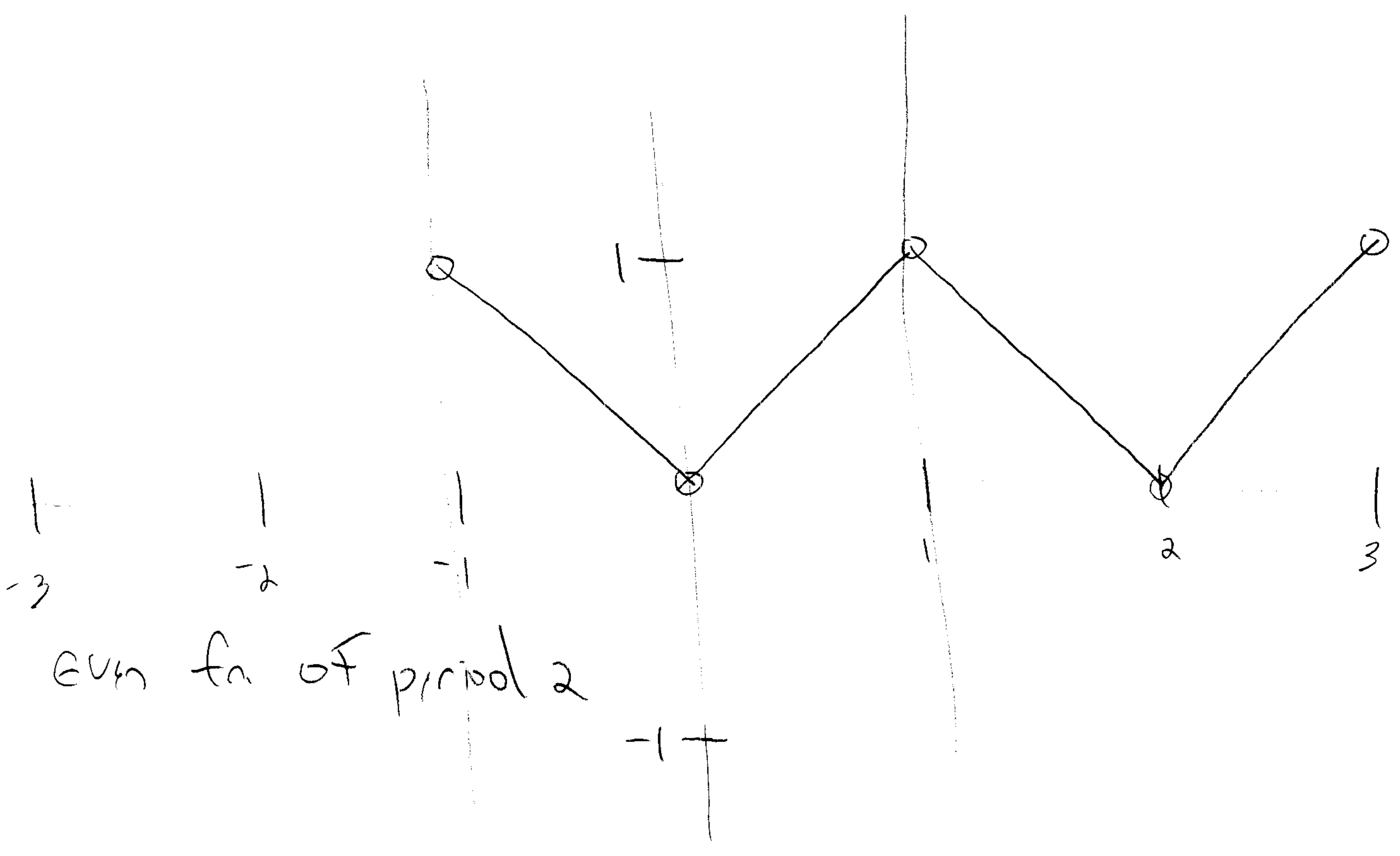
and your function depends on z now

1.9)

15.) $f(x) = x \quad 0 < x < 1$



odd fn of period 2



even fn of period 2

$\underline{\underline{\text{odd}}}$ $f(x) = x \quad -1 < x < 1 \qquad l = 1$

$a_n = 0$

$$b_n = \frac{2}{1}\int_0^1 x \sin \frac{n\pi x}{1}\, dx$$

$$(x \cos n\pi x)' = \cos n\pi x - n\pi\, x \sin n\pi x$$ have

$$= \frac{2}{\pi n}\left[- \Big|_0^1 x \cos n\pi x + \int_0^1 \cos n\pi x \right]$$ $\int_0^1 \cos n\pi x \to 0$

$$= \frac{2}{\pi n}\left[-\cos n\pi \right] = \frac{2}{n\pi}(-1)^{n+1}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$= \frac{2}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \dots \right)$$

even $\quad f(x) = \begin{cases} x & 0 < x < 1 \\ -x & 0 < x < -1 \end{cases}$

$$a_0 = 2\int_0^1 x = 2 \cdot \frac{1}{2} = 1$$

$$\underset{n \neq 0}{a_n} = 2\int_0^1 x \cos n\pi x$$

$$(x \sin n\pi x)' = \sin n\pi x + n\pi \overset{\text{have}}{x \cos n\pi x}$$

$$= \frac{2}{n\pi}\left[\cancel{\Big|_0^1 x \sin n\pi x}^{\,0} - \int_0^1 \sin n\pi x \right]$$

$$= \frac{2}{n\pi}\left[\frac{1}{n\pi}\Big|_0^1 \cos n\pi x \right] = \frac{2}{n^2\pi^2}\left[\cos n\pi - 1 \right]$$

$$= \frac{-4}{n^2\pi^2} \quad n \text{ odd}$$

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\cos \pi x + \frac{\cos 3\pi x}{9} + \dots \right]$$

1) Use $f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x)\right]}_{\text{even fn.}} + \underbrace{\frac{1}{2}\left[f(x) - f(-x)\right]}_{\text{odd fn.}}$

a) $e^{inx} = \frac{1}{2}\left[e^{inx} + e^{-inx}\right] + \frac{1}{2}\left[e^{inx} - e^{-inx}\right]$

$= \cos nx + i \sin x$

b) $xe^{x} = \frac{1}{2}\left[xe^{x} + -xe^{-x}\right] + \frac{1}{2}\left[xe^{x} - -xe^{-x}\right]$

$= x\left[\frac{e^{x} - e^{-x}}{2}\right] + x\left[\frac{e^{x} + e^{-x}}{2}\right]$

$= x \sinh x + x \cosh x$

note: $x \sinh x$ is odd since

x is an odd fn. and $\sinh x$ is an odd fn.

thus their product will be even

23. We want a Fourier sine series to represent a function given on $(0,\ell)$. We must then extend the function to be odd on $(-\ell,\ell)$ and continue it periodically with period 2ℓ. Note, however, from text equation (9.4) that we actually use only the values of the function on $(0,\ell)$ in computing the coefficients. From the text figure, we have

$$f(x,0)=\begin{cases}\frac{2h}{\ell}x, & 0<x<\frac{\ell}{2},\\ \frac{2h}{\ell}(\ell-x), & \frac{\ell}{2}<x<\ell .\end{cases}$$

Then by text equation (9.4):

$$b_n=\frac{2}{\ell}\left[\int_0^{\ell/2}\frac{2h}{\ell}x\sin\frac{n\pi x}{\ell}\,dx+\int_{\ell/2}^{\ell}\frac{2h}{\ell}(\ell-x)\sin\frac{n\pi x}{\ell}\,dx\right]$$

$$=\frac{4h}{\ell^2}\frac{\ell^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{\ell}-\frac{n\pi x}{\ell}\cos\frac{n\pi x}{\ell}\right)\Bigg|_0^{\ell/2}-\frac{4h}{\ell}\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\Bigg|_{\ell/2}^{\ell}$$

$$-\frac{4h}{\ell^2}\frac{\ell^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{\ell}-\frac{n\pi x}{\ell}\cos\frac{n\pi x}{\ell}\right)\Bigg|_{\ell/2}^{\ell}$$

$$=\frac{4h}{n^2\pi^2}\left(\sin\frac{n\pi}{2}-\frac{n\pi}{2}\cos\frac{n\pi}{2}\right)-\frac{4h}{n\pi}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)$$

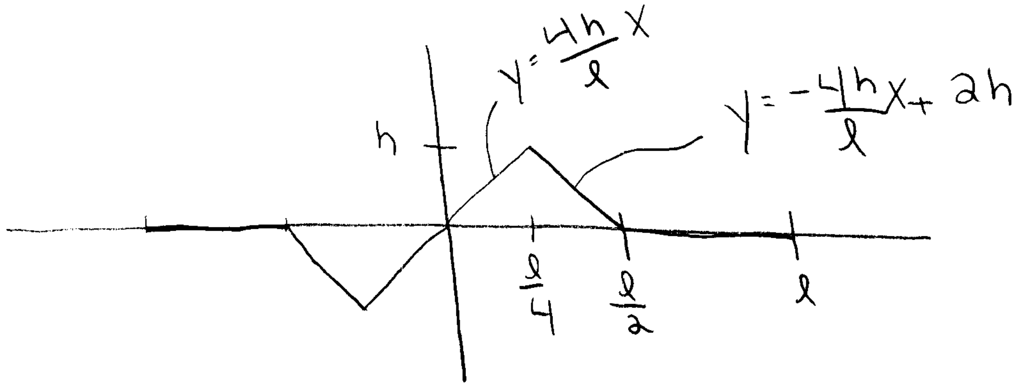
$$-\frac{4h}{n^2\pi^2}\left(\sin n\pi-n\pi\cos n\pi-\sin\frac{n\pi}{2}+\frac{n\pi}{2}\cos\frac{n\pi}{2}\right)$$

$$=\frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}=\frac{8h}{n^2\pi^2}\begin{cases}1, & n=1+4k,\\ 0, & n\text{ even},\\ -1, & n=3+4k.\end{cases}$$

$$f(x)=\frac{8h}{\pi^2}\left(\sin\frac{\pi x}{\ell}-\frac{1}{9}\sin\frac{3\pi x}{\ell}+\frac{1}{25}\sin\frac{5\pi x}{\ell}\cdots\right).$$

24.) extend our fn. to $(-\ell, 0)$

and make it odd



this has a period of 2ℓ

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx$$

$$= \frac{2}{\ell} \left[\int_0^{\frac{\ell}{4}} \frac{4h}{\ell} x \sin \frac{n\pi x}{\ell}\, dx + \int_{\frac{\ell}{4}}^{\frac{\ell}{2}} \left(-\frac{4h}{\ell} x + 2h \right) \sin \frac{n\pi x}{\ell}\, dx \right]$$

(let $u = \frac{n\pi x}{\ell}$)

$$= \frac{2}{\ell} \left[\frac{4h}{\ell} \left(\frac{\ell}{n\pi} \right)^2 \int_{0}^{\frac{\ell}{4}} u \sin u\, du + \frac{-4h}{\ell} \left(\frac{\ell}{n\pi} \right)^2 \int_{\frac{\ell}{4}}^{\frac{\ell}{2}} u \sin u\, du \right.$$

$$\left. + \frac{2h\,\ell}{n\pi} \int_{\frac{\ell}{4}}^{\frac{\ell}{2}} \sin u\, du \right]$$

$$\int u \sin u \, du = \sin u - u \cos u$$

$$\int \sin u \, du = -\cos u$$

$$= \frac{2}{\ell}\left[\frac{4h}{\ell}\left(\frac{\ell}{n\pi}\right)^2 \left[\sin \frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{4} - \frac{\ell}{4} \cos \frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{4} \right] \right.$$

$$+ \frac{4h}{\ell}\left(\frac{\ell}{n\pi}\right)^2 \left[-\sin \frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{2} + \frac{\ell}{2} \cos \frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{2} + \sin \frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{4} - \frac{\ell}{4} \cos \frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{4} \right]$$

$$\left. + 2h \frac{\ell}{n\pi} \left[-\cos \frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{2} + \cos \frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{4} \right] \right]$$

b_n

$$= \frac{2}{\ell}\left[\frac{8h}{\ell}\left(\frac{\ell}{2\pi}\right)^2 \left[\sin \frac{n\pi}{4} - \frac{\ell}{4} \cos \frac{n\pi}{4} \right] \right.$$

$$+ \frac{4h}{\ell}\left(\frac{\ell}{n\pi}\right)^2 \left[\frac{\ell}{2} \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} \right]$$

$$\left. + 2h \left(\frac{\ell}{n\pi}\right) \left[\cos \frac{n\pi}{4} - \cos \frac{n\pi}{2} \right] \right]$$