# HW 3

## 7.11

2) show 11.5

average value of $|f(x)|^2$ (over a period)

$$= \frac{1}{2\pi}\int \sum_{n=-\infty}^{\infty} c_n e^{inx} \sum_{j=-\infty}^{\infty} \overline{c_j} e^{-ijx}\, dx$$

$$= \frac{1}{2\pi} \sum_n \sum_j \int c_n \overline{c_j}\, e^{inx} e^{-ijx}$$

$$(=0 \text{ if } j \neq n)$$

$$= \frac{1}{2\pi} \sum_n c_n \overline{c_n}\, 2\pi$$

$$= \sum_n |c_n|^2$$

note: if $f(x)$ is real

$$f(x) - \overline{f(x)} = 0$$

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} - \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-inx} = 0$$

$$\sum_{n=-\infty}^{\infty} (c_n - \overline{c_{-n}}) e^{inx} = 0$$ (multiply each side by $e^{inx}$ and integrate over a period)

$$c_n = \overline{c_{-n}}$$

7.12

14.) a) $f(x) = (x-1)^2$ on $(0,2)$ has period 2

$$C_0 = \frac{1}{2}\int_0^2 (x-1)^2\,dx = \frac{1}{2}\Big|_0^2 \frac{x^3}{3} - \cancel{2}\frac{x^2}{\cancel{2}} + x = \frac{1}{2}\left(\frac{8}{3} - 4 + 2\right) = \frac{1}{3}$$

$$C_n = \frac{1}{2}\int_0^2 (x-1)^2 e^{-in\pi x}\,dx$$

0 has period of 2

$$= \frac{1}{2}\Big[\underbrace{\int_0^2 x^2 e^{-in\pi x}}_{A} - 2\underbrace{\int_0^2 x e^{-in\pi x}}_{B} + \cancel{\int_0^2 e^{-in\pi x}}\Big]$$

for B:

will $\int$ to 0

$$(x e^{-in\pi x})' = \cancel{e^{-in\pi x}} - in\pi\, x e^{-in\pi x}$$ have

$$\int_0^2 x e^{-in\pi x} = \frac{-1}{in\pi}\Big|_0^2 x e^{-in\pi x} = \frac{-1}{in\pi}\, 2e^{-in2\pi} = \frac{-2}{in\pi}$$

for (A)

$$\left(x^2 e^{-in\pi x}\right)' = 2x e^{-in\pi x} - in\pi\, x^2 e^{-in\pi x} \quad \text{new}$$

$$\int_0^2 x^2 e^{-in\pi x} = \frac{1}{in\pi}\left[-\Big|_0^2 x^2 e^{-in\pi x} + 2\int x e^{-in\pi x}\right]$$

from (A)

$$= \frac{1}{in\pi}\left[-4 e^{-in2\pi} + 2\left(\frac{-2}{in\pi}\right)\right]$$

$$= \frac{1}{in\pi}\left[-4 - \frac{4}{in\pi}\right]$$

so

$$C_n = \frac{1}{2}\left[\cancel{\frac{-4}{in\pi}} - \frac{4}{(in\pi)^2} + \cancel{\frac{4}{in\pi}}\right]$$

$$C_n = \frac{2}{n^2\pi^2}$$

b) use

average value of $|f(x)|^2$ (over 1 period)

$$= \sum_{-\infty}^{\infty} |c_n|^2$$

so $\quad \text{avg. } |f(x)|^2 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left|\frac{2}{n^2\pi^2}\right|^2 + \frac{1}{9}$

$$= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{4}{n^4\pi^4} + \frac{1}{9}$$

$$= \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{1}{9}$$

so $\quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{8}\left(\text{avg. } |f(x)|^2 - \frac{1}{9}\right)$

and $\quad \text{avg. } |f(x)|^2 = \dfrac{\int_0^2 |f(x)|^2 \, dx}{2}$

$$\int_0^2 (x-1)^2(x-1)^2\,dx = \int_0^2 (x-1)^4\,dx$$

let $u = x-1$, $du = dx$

$$= \int_{0}^{2} u^4\,du = \Big|_0^2 \frac{(x-1)^5}{5} = \frac{(2-1)^5}{5} - \frac{(0-1)^5}{5} = \frac{1}{5} - -\frac{1}{5}$$

$$= \frac{2}{5}$$

thus $$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{8}\left(\frac{2}{10} - \frac{1}{9}\right) = \frac{\pi^4}{\cancel{8}} \cdot \frac{\cancel{8}}{90} = \frac{\pi^4}{90}$$

13.1) $\nabla \cdot D = \rho$

$D = -\varepsilon \nabla \phi$

0 if $\varepsilon$ is indeppendent of position

$\nabla \cdot -\varepsilon(\nabla\phi) = -\varepsilon(\nabla \cdot \nabla\phi) + \nabla\phi \cdot \cancel{\nabla}(-\varepsilon)$

$= -\varepsilon \nabla^2 \phi = \rho$

$\nabla^2 \phi = \frac{-\rho}{\varepsilon}$

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if region is charge free $\rho = 0$

Section 2

1. The basic solutions of Laplace's equation are given by text equation (2.7). We want $T \to 0$ as $y \to \infty$ and $T = 0$ when $x = 0$, so we use the solution $e^{-ky} \sin kx$. We also want $T = 0$ when $x = 10$ so we set $\sin 10k = 0$ or $10k = n\pi$. Thus our basic solutions are $e^{-n\pi y/10} \sin(n\pi x/10)$, $n = 1,2,3 \cdots$. Any linear combination of these solutions is a solution of Laplace's equation satisfying the given boundary conditions on three sides of the semi-infinite strip. We write

(1) $\quad T = \sum b_n e^{-n\pi y/10} \sin(n\pi x/10).$

Now we want $T = x$ when $y = 0$:

(2) $\quad T = x = \sum b_n \sin(n\pi x/10).$

This means that we want to expand x in a Fourier sine series. By text Chapter 7, Section 9, we find

$$b_n = \frac{2}{10}\int_0^{10} x \sin\frac{n\pi x}{10}\,dx = \frac{2}{10}\left(\frac{10}{n\pi}\right)^2 \left(\sin\frac{n\pi x}{10} - \frac{n\pi x}{10}\cos\frac{n\pi x}{10}\right)\Bigg|_0^{10}$$

$$= \frac{20}{n^2\pi^2}(-n\pi\cos n\pi) = -\frac{20}{n\pi}(-1)^n.$$

We substitute the values of $b_n$ into (1) [caution: not into (2), which is just a step in our work and not the final answer] to obtain $T(x,y)$ satisfying Laplace's equation and all the boundary conditions:

$$T(x,y) = \frac{20}{\pi}\sum \frac{(-1)^{n+1}}{n} e^{-n\pi y/10} \sin\frac{n\pi x}{10}.$$

Further comment: Note that we can now easily find the temperature distribution in a finite plate. Suppose that we cut the semi-infinite plate off at height 15 cm and keep the top edge at 0°. Then we replace (1) above by

(3) $\quad T = \sum B_n \sinh\frac{n\pi}{10}(15 - y) \sin\frac{n\pi x}{10}$

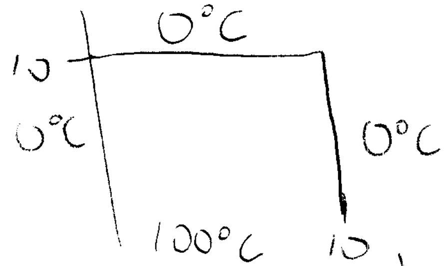
so that $T = 0$ at $y = 15$ as well as at $x = 0$ and $x = 10$ (see text, page 546). Then at $y = 0$, we want

(4) $\quad T = x = \sum B_n \sinh\frac{15n\pi}{10} \sin\frac{n\pi x}{10} = \sum b_n \sin\frac{n\pi x}{10}$

where $B_n \sinh\frac{3n\pi}{2} = b_n$. The Fourier coefficients $b_n$ are the same as above. We solve for $B_n$ and substitute into (3) to find the temperature distribution in the finite plate:

$$T = \frac{20}{\pi}\sum \frac{(-1)^{n+1}}{n \sinh(3n\pi/2)} \sinh\frac{n\pi}{10}(15 - y) \sin\frac{n\pi x}{10}.$$

$\frac{13.2}{10.1}$




for the left-hand side at $X = 0$

discard the solutions that contain $\cos kx$

since $\cos 0 = 1$

for the right-hand side at $X = 10$

we have $\sin k10 = 0$

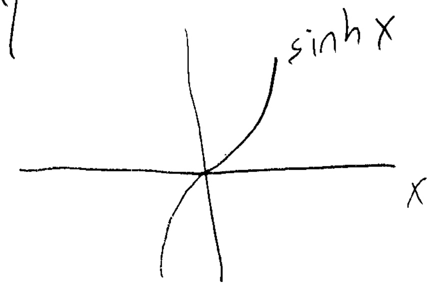
which implies $k10 = n\pi$ $\quad n \in \mathbb{Z}$ (integers)

$k = \frac{n\pi}{10}$

for the $y$ direction we can have a linear combo

of $e^{ky}$ and $e^{-ky}$

choose $\frac{e^{ky} - e^{-ky}}{2} = \sinh ky$




since $\sinh 0 = 0$

but we must shift this to match our b.c. at $y=10$

use $\sinh k(10-y)$

to satisfy the b.c. at $y=0$

use a linear combo of solutions of

$$T = \sinh\left(\frac{n\pi}{10}(10-y)\right)\sin\left(\frac{n\pi}{10}x\right)$$

$$\text{so } T = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi}{10}(10-y)\right)\sin\left(\frac{n\pi}{10}x\right)$$

at $y=0$ $T=100$

$$100 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}x\right) \qquad B_n = b_n \sinh n\pi$$

fourier sine series for $f(y)=100$

make it an even fn. in $(-10,10)$

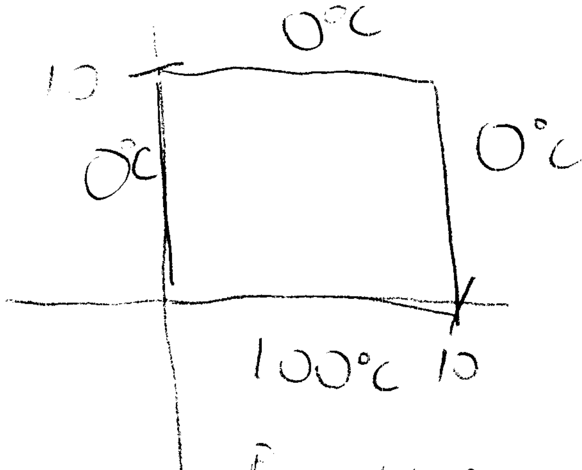
$$B_n = \frac{2}{10}\int_0^{10} 100 \sin\frac{n\pi x}{10}\,dx = \frac{-2}{\cancel{10}}\cdot 100 \cdot \frac{\cancel{10}}{n\pi}\Big|_0^{10} \cos\frac{n\pi x}{10}$$

$$= \frac{-200}{n\pi}\left(\cos n\pi - 1\right) = \frac{400}{n\pi} \quad n \text{ odd}$$

$$\text{so } T = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{400}{n\pi \sinh(n\pi)} \sinh\left(\frac{n\pi}{10}(10-y)\right)\sin\left(\frac{n\pi}{10}x\right)$$

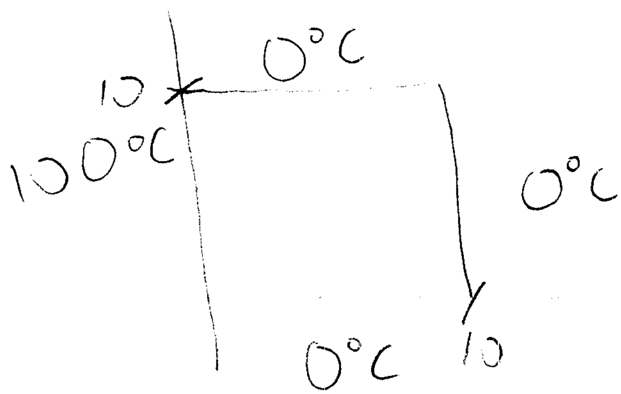
13.2

11.)



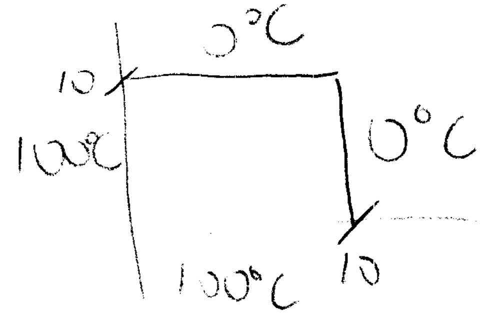

from 10) $T_1 = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{400}{n\pi \sinh(n\pi)} \sinh\left(\frac{n\pi}{10}(10-y)\right) \sin\left(\frac{n\pi}{10} x\right)$

now for




ump $T_2 = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{400}{n\pi \sinh(n\pi)} \sinh\left(\frac{n\pi}{10}(10-x)\right) \sin\left(\frac{n\pi}{10} y\right)$

thus for




$T = T_1 + T_2$