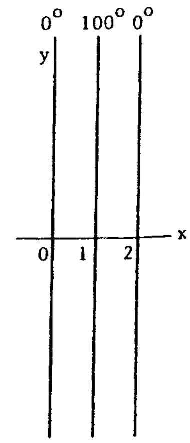
Section 3

5. Initially the temperatures are as shown in the diagram. For each slab, the temperature is linear, $u_0 = ax + b$, where a and b must be found for each slab.



For the first slab:

$u_0 = 0$ when $x = 0$ and $u_0 = 100$ when $x = 1$.

For the second slab:

$u_0 = 100$ when $x = 1$ and $u_0 = 0$ when $x = 2$.

Thus the initial temperature distribution is

$$u_0 = \begin{cases} 100x, & 0 < x < 1, \\ 100(2 - x), & 1 < x < 2. \end{cases}$$

The final temperature is $u_f = 100$. The temperature distribution as a function of x and t must be some linear combination of the basic solutions of the heat flow equation [text equation (3.1)]. These solutions are given by text equation (3.10) when $k > 0$. When $k = 0$, text equations (3.5), (3.6), (3.8) and (3.9) become

$$\nabla^2 F = 0 \quad \text{or} \quad \frac{d^2F}{dx^2} = 0, \quad F = ax + b,$$

$$\frac{dT}{dt} = 0, \quad T = \text{const.}$$

Thus the basic solutions of the heat flow equation are

$$u = \begin{cases} e^{-k^2\alpha^2 t} \sin kx, & k > 0, \\ e^{-k^2\alpha^2 t} \cos kx, & k > 0, \\ ax + b, & k = 0, \end{cases}$$

and we can write the solution of our problem in the form

$$(1) \qquad u = \sum_k e^{-k^2\alpha^2 t}(b_k \sin kx + a_k \cos kx) + ax + b.$$

As $t \to \infty$, we see from (1) that $u \to ax + b$; this must be the final steady state u_f. In our problem $u_f = 100$ so we can write (1) as

$$(2) \qquad u = \sum_k e^{-k^2\alpha^2 t}(b_k \sin kx + a_k \cos kx) + 100.$$

Now we must satisfy the conditions $u = 100$ at $x = 0$ and at $x = 2$ for all t. From (2) we see that $u = 100$ if the terms in the series are zero. This will be true for all t if we keep only the sine terms and take $k = n\pi/2$. Thus we write (2) as

$$(3) \qquad u = 100 + \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/2)^2 t} \sin\frac{n\pi x}{2}.$$

[This is text equation (3.16) with $\ell = 2$ and $u_f = 100$.]

When $t = 0$, (3) becomes

$$u_0 = 100 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \qquad \text{or}$$

$$(4) \qquad u_0 - 100 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}.$$

Equation (4) says to expand in a Fourier sine series the function

$$u_0 - u_f = \begin{cases} 100x - 100 = 100(x-1), & 0 < x < 1, \\ 100(2-x) - 100 = -100(x-1), & 1 < x < 2. \end{cases}$$

We find

$$\frac{b_n}{100} = \frac{2}{2}\left[\int_0^1 (x-1)\sin\frac{n\pi x}{2}\,dx - \int_1^2 (x-1)\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \left[\left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi x}{2} - \frac{2}{n\pi}(x-1)\cos\frac{n\pi x}{2}\right]_0^1 - \left[\begin{matrix}\text{same}\\ \text{integral}\end{matrix}\right]_1^2$$

$$= \left(\frac{2}{n\pi}\right)^2\left(2\sin\frac{n\pi}{2}\right) + \frac{2}{n\pi}(-1 + \cos n\pi).$$

$$b_n = 100\begin{Bmatrix} 0, & \text{even } n \\ \dfrac{8}{n^2\pi^2} - \dfrac{4}{n\pi}, & n = 1 + 4k \\ -\dfrac{8}{n^2\pi^2} - \dfrac{4}{n\pi}, & n = 3 + 4k \end{Bmatrix} = 400\begin{Bmatrix} 0, & \text{even } n, \\ \dfrac{2}{n^2\pi^2} - \dfrac{1}{n\pi}, & n = 1 + 4k, \\ -\dfrac{2}{n^2\pi^2} - \dfrac{1}{n\pi}, & n = 3 + 4k. \end{Bmatrix}$$

Then the temperature distribution is given by (3) above with these values for the b_n.

8.) using Dr. Saito's comments

- the final steady-state solution satisfys Laplace's eq. $\nabla^2 u_f = 0$

which gives possible solutions of the form $U_f(x) = ax + b$

for $t > 0$, $x = 0$ is at $0°$

$x = 2$ is at $100°$

so $a = 50$ and $b = 0$

so $U_f(x) = 50x$

- since $\tilde{u}(0,t) = \tilde{u}(\ell,t) = 0$ for all $t > 0$

you must discard the solutions that contain $\cos kx$

also it restricts k

$\sin 2k = 0$

$2k = n\pi$

$k = \frac{n\pi}{2}$ $n = 1, 2, 3, \ldots$

- lastly $U_0(x) = 0$

so $\tilde{U}(x,0) = U_0(x) - U_E(x) = -50x$

we satisfy this by summing over all possible solutions

$$\tilde{U}(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{2}\right)^2 \alpha^2 t} \sin \frac{n\pi}{2} x$$

$$\tilde{U}(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} = -50x$$

$$b_n = \frac{2}{2} \int_0^2 (-50x) \sin \frac{n\pi x}{2} \, dx$$

$$= -50 \int_0^2 x \sin \frac{n\pi x}{2} \, dx$$

$$\left(x \cos \frac{n\pi x}{2}\right)' = \cos \frac{n\pi x}{2} - \frac{n\pi}{2} x \sin \frac{n\pi x}{2}$$

$$= -50 \cdot \frac{2}{n\pi} \left[\underbrace{\int_0^2 \cos \frac{n\pi x}{2}}_{0} - \Big[x \cos \frac{n\pi x}{2} \Big]_0^2 \right]$$

$$= \frac{-100}{n\pi} \cdot -2\cos\frac{n\pi\cancel{2}}{\cancel{2}}$$

$$= \frac{200(-1)^n}{n\pi}$$

thus $$U(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{2}\right)^2 \alpha^2 t} \sin\frac{n\pi x}{2} + 50x$$

$$b_n = \frac{200(-1)^n}{n\pi}$$

10.) $\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$

let $u = X(\bar{x}) T(t)$

$$T \nabla^2 X = \frac{1}{v^2} X \frac{\partial^2 T}{\partial t^2}$$

divide by XT

$$\frac{1}{X} \nabla^2 X = \frac{1}{v^2} \frac{1}{T} \frac{\partial^2 T}{\partial t^2}$$

the lhs is a fn. of only space variables
while the rhs is a fn. of only time

set both equal to the constant $-k^2$

$$\nabla^2 X = -k^2 X \qquad \frac{\partial^2 T}{\partial t^2} = -v^2 k^2 T$$

$$\nabla^2 X + k^2 X = 0 \qquad \frac{\partial^2 T}{\partial t^2} + v^2 k^2 T = 0$$

2.) - since the string is fastened at $x=0, l$ for all time at $y=0$, discard solutions containing $\cos kx$

- since the velocity of all the points on the string at $t=0$ is 0, discard solutions containing $\sin \omega t$

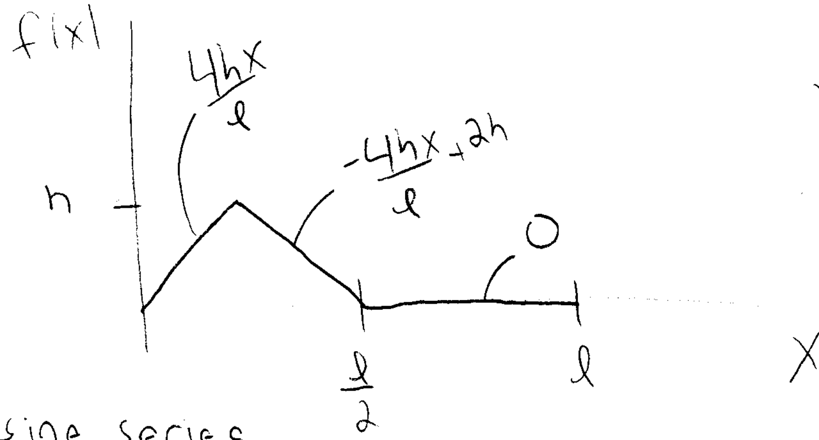
- for all time we want $y=0$ at $x=0$ and $x=l$
this restricts k

$$\sin kl = 0$$

$$\rightarrow k = \frac{n\pi}{l} \qquad n = 1, 2, 3, \ldots$$

- lastly we need y to equal the displacement of the string at $t=0$
so

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi v t}{l}$$



$y(x,0) = f(x)$

a fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\, f(x)\, dx$$

$$= \frac{2}{l}\left[\frac{4h}{l}\int_0^{\frac{l}{4}} x\sin\frac{n\pi x}{l} - \frac{4h}{l}\int_{\frac{l}{4}}^{\frac{l}{2}} x\sin\frac{n\pi x}{l} + 2h\int_{\frac{l}{4}}^{\frac{l}{2}} \sin\frac{n\pi x}{l}\right]$$

$$\left(x\cos\frac{n\pi x}{l}\right)' = \cos\frac{n\pi x}{l} - \frac{n\pi}{l}x\sin\frac{n\pi x}{l}$$

$$\int x\sin\frac{n\pi x}{l} = \frac{l}{n\pi}\int \cos\frac{n\pi x}{l} - \frac{l}{n\pi}\Big|\, x\cos\frac{n\pi x}{l}$$

so $$b_n = \frac{2}{\ell}\left[\frac{\cancel{\ell}4h}{n\pi\cancel{\ell}}\left[\frac{\ell}{n\pi}\Big|_0^{\frac{\ell}{4}}\sin\frac{n\pi x}{\ell} - \Big|_0^{\frac{\ell}{4}} x\cos\frac{n\pi x}{\ell}\right]\right.$$

$$-\frac{4h}{n\pi}\left[\frac{\ell}{n\pi}\Big|_{\frac{\ell}{4}}^{\frac{\ell}{2}}\sin\frac{n\pi x}{\ell} - \Big|_{\frac{\ell}{4}}^{\frac{\ell}{2}} x\cos\frac{n\pi x}{\ell}\right]$$

$$\left.-\frac{2h\ell}{n\pi}\Big|_{\frac{\ell}{4}}^{\frac{\ell}{2}}\cos\frac{n\pi x}{\ell}\right]$$

$$=\frac{2}{\ell}\left[\frac{4h}{n\pi}\left[\frac{\ell}{n\pi}\sin\frac{n\pi\cancel{\ell}}{\cancel{\ell}4} - \cancel{\frac{\ell}{4}\cos\frac{n\pi\ell}{\ell 4}}\right]\right.$$

$$-\frac{4h}{n\pi}\left[\frac{\ell}{n\pi}\left(\sin\frac{n\pi\cancel{\ell}}{\cancel{\ell}2} - \sin\frac{n\pi\cancel{\ell}}{\cancel{\ell}4}\right) - \cancel{\frac{\ell}{2}\cos\frac{n\pi\ell}{\ell 2}} + \cancel{\frac{\ell}{4}\cos\frac{n\pi\ell}{\ell 4}}\right]$$

$$\left.-\frac{2h\ell}{n\pi}\left[\cancel{\cos\frac{n\pi\ell}{\ell 2}} - \cancel{\cos\frac{n\pi\ell}{\ell 4}}\right]\right]$$

$$=\frac{2}{\ell}\left[\frac{8h\ell}{(n\pi)^2}\sin\frac{n\pi}{4} - \frac{4h\ell}{(n\pi)^2}\sin\frac{n\pi}{2}\right] = b_n$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}\cos\frac{n\pi v}{\ell}t$$

5.) - ends of string fixed for all t at $y=0$

discard solutions containing $\cos kx$

- at $t=0$, the points on the string are given an initial velocity

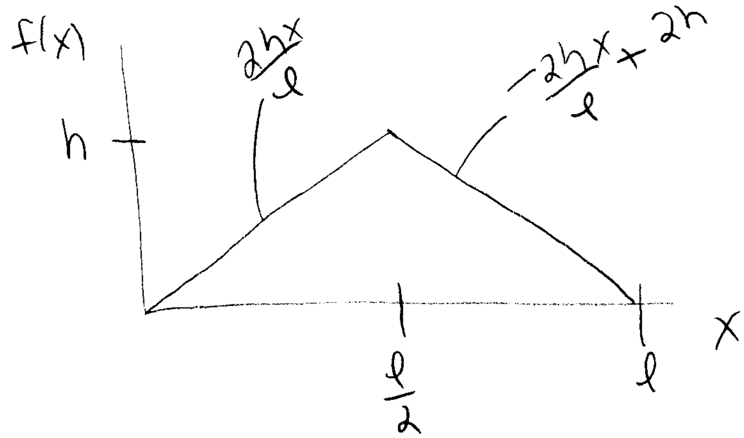
discard solutions containing $\cos \omega t$

- for all time we want $y=0$ at $x=0$ and $x=l$

$$\sin kl = 0$$

$$k = \frac{n\pi}{l} \qquad n = 1, 2, \ldots$$

- We need to match the initial velocity given to points on the string



$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{n\pi v t}{l}$$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{l} \sin \frac{n\pi x}{l} \cos \frac{n\pi v t}{l}$$

$$\left(\frac{dy}{dt}\right)_{t=0} = \frac{\pi v}{l} \sum_{n=1}^{\infty} b_n n \sin \frac{n\pi x}{l} = f(x)$$

let $C_n = b_n n$

so we have

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = \frac{l}{\pi v} f(x)$$

so $$C_n = \frac{l}{\pi v} \left(\frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \right)$$

this was solved in HW2 problem 23

$$C_n = \frac{l}{\pi v} \left(\frac{8h}{n^2\pi^2} \sin \frac{n\pi}{2} \right)$$

$$b_n = \frac{8hl}{v n^3 \pi^3} \sin \frac{n\pi}{2}$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{n\pi v t}{l}$$

9.) (for 2,5)

We look for the characteristic frequency for all x whose amplitude has the greatest magnitude

for 2,5 $$\omega_n = \frac{n\pi v}{l}$$

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{nv}{2l}$$

for 2.) $$B_1 = \frac{2\sin\frac{\pi}{4} - \sin\frac{\pi}{2}}{1} = \sqrt{2} - 1 \approx 0.41$$

$$B_n = \frac{2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2}}{n^2}$$

$$B_2 = \frac{2\sin\frac{\pi}{2} - \sin\pi}{4} = \frac{1}{2}$$

$$B_3 = \frac{2\sin\frac{3\pi}{4} - \sin\frac{3\pi}{2}}{9} = \frac{\sqrt{2} - -1}{9} = \frac{\sqrt{2}+1}{9} \approx 0.27$$

the absolute value of the numerator will be bounded by 3 and the denominator only will grow

so choose $n = 2$

$$\nu_2 = \frac{\cancel{2}v}{\cancel{2}l} = \frac{v}{l}$$

5.)

$$b_n = \frac{8hl}{v n^3 \pi^3} \sin \frac{n\pi}{2}$$

$$b_1 = \frac{8hl}{v \pi^3}$$

$$b_2 = 0$$

$$b_3 = -\frac{8hl}{v 27 \pi^3}$$

$$b_4 = 0$$

again thy absoluty valuy of the numerator is bounded and thy denominator only will grow

choose $n = 1$

$$V_1 = \frac{V}{2l}$$

4.) L3

$$\int_0^\infty \sin at\, e^{-pt}\, dt = \frac{a}{p^2+a^2} \qquad \mathrm{Re}\, p > |\mathrm{Im}\, a|$$

differentiate with respect to a

$$\int_0^\infty t \cos at\, e^{-pt}\, dt = \frac{(p^2+a^2)a' - a(p^2+a^2)'}{(p^2+a^2)^2}$$

$$= \frac{p^2+a^2-2a^2}{(p^2+a^2)^2}$$

$$= \frac{p^2-a^2}{(p^2+a^2)^2}$$

L12

6.) L2

$$\int_0^\infty e^{-at} e^{-pt}\, dt = \frac{1}{p+a} \qquad \mathrm{Re}(p+a) > 0$$

let $a \longrightarrow a+bi$

$$\int_0^\infty e^{-(a+bi)t} e^{-pt}\, dt = \frac{1}{p+a+bi} \qquad (1)$$

let $a \longrightarrow a-bi$

$$\int_0^\infty e^{-(a-bi)t} e^{-pt}\, dt = \frac{1}{p+a-bi} \qquad (2)$$

note:

in (1) $\quad e^{-(a+bi)t} = e^{-at} e^{-ibt} = e^{-at}(\cos bt - i \sin bt)$

in (2) $\quad e^{-(a-bi)t} = e^{-at} e^{ibt} = e^{-at}(\cos bt + i \sin bt)$

so

(1) + (2)

$$\rightarrow \int_0^\infty 2e^{-at}\cos bt\, e^{-pt}\,dt = \frac{1}{p+a+bi} + \frac{1}{p+a-bi}$$

$$= \frac{p+a-\cancel{bi}+p+a+\cancel{bi}}{p^2+2ap+a^2+b^2}$$

$$= \frac{2p+2a}{(p+a)^2+b^2}$$

$$\int_0^\infty e^{-at}\cos bt\, e^{-pt}\,dt = \frac{p+a}{(p+a)^2+b^2} \qquad (L14)$$

(2) - (1)

$$\rightarrow \int_0^\infty 2i\,e^{-at}\sin bt\, e^{-pt}\,dt = \frac{1}{p+a-bi} - \frac{1}{p+a+bi}$$

$$= \frac{2bi}{(p+a)^2+b^2}$$

$$\int_0^\infty e^{-at}\sin bt\, e^{-pt}\,dt = \frac{b}{(p+a)^2+b^2} \qquad (L13)$$

10.) $\dfrac{2p-1}{p^2-2p+10} = \dfrac{2p-1}{(p-1)^2+3^2} = 2\left(\dfrac{p}{(p-1)^2+3^2}\right) - \left(\dfrac{1}{(p-1)^2+3^2}\right)$

now $L(e^{-at}\sin bt) = \dfrac{b}{(p+a)^2+b^2}$

$L(e^{-at}\cos bt) = \dfrac{p+a}{(p+a)^2+b^2}$

so

$$= 2\left(\frac{p-1+1}{(p-1)^2+3^2}\right) - \left(\frac{1}{(p-1)^2+3^2}\right) = 2\left(\frac{p-1}{(p-1)^2+3^2}\right) + \frac{1}{3}\left(\frac{3}{(p-1)^2+3^2}\right)$$

so $L^{-1}\left(\dfrac{2p-1}{p^2-2p+10}\right) = 2e^{t}\cos 3t + \dfrac{1}{3}e^{t}\sin 3t$

17) Want $L(t^2 \sin at)$

(L11) $\int t \sin at \, e^{-pt} \, dt = \frac{2ap}{(p^2+a^2)^2}$

(L32) $\int t^n g(t) \, e^{-pt} \, dt = (-1)^n \frac{d^n G(p)}{dp^n}$

if we let $n=1$ and $g(t) = t \sin at$ in L32 we will find that

$$L(t^2 \sin at) = (-1)^1 \frac{d}{dp}\left(\frac{2ap}{(p^2+a^2)^2}\right)$$

$$= -\left(\frac{(p^2+a^2)^2 \cdot 2a - 2ap \cdot 2(p^2+a^2) \cdot 2p}{(p^2+a^2)^4}\right)$$

$$= \frac{6ap^2 - 2a^3}{(p^2+a^2)^3}$$

19.) $$L f(t) = \int_0^{\infty} f(t) e^{-pt}\, dt = F(p) \quad (2.1)$$

now $$L\left(e^{-at} g(t)\right) = \int_0^{\infty} e^{-at} g(t) e^{-pt}\, dt$$

$$= \int_0^{\infty} g(t) e^{-(p+a)t}\, dt$$

$$= G(p+a)$$