HW5

15.3

5.) $y'' + y = \sin t \quad y_0 = 0, \ y_0' = -\frac{1}{2}$

i) take the Laplace transform of both sides

$L\, y'' + L\, y = L \sin t$

$$p^2 Y - p y_0 - y_0' + Y = \frac{1}{p^2+1}$$

by L35
L3 with a=1
and $Y = L(y)$

ii) plug in the initial conditions

$$p^2 Y + \frac{1}{2} + Y = \frac{1}{p^2+1}$$

iii) solve for Y as a fn of p

$$Y = \left(\frac{1}{p^2+1} - \frac{1}{2}\right) \cdot \frac{1}{(p^2+1)} = \frac{1}{(p^2+1)^2} - \frac{1}{2}\frac{1}{(p^2+1)}$$

iv) take the inverse transform of both sides to get y as a function of t

$$L^{-1} Y = L^{-1}\left(\frac{1}{(p^2+1)^2}\right) - \frac{1}{2} L^{-1}\left(\frac{1}{p^2+1}\right)$$

$\downarrow$ L17 $\qquad \downarrow$ L3

$$y(t) = \frac{1}{2}(\sin t - t\cos t) - \frac{1}{2}\sin t = -\frac{1}{2} t\cos t$$

25.) $y'' + 4y' + 5y = 2e^{-2t}\cos t \qquad y_0 = 0,\ y_0' = 3$

$$L\,y'' + 4L\,y' + 5L\,y = 2L\left(e^{-2t}\cos t\right)$$

use $L\,35$

$Ly = Y$

$L\,14$ with $a = 2,\ b = 1$

$$p^2Y - py_0 - y_0' + 4pY - 4y_0 + 5Y = 2\left(\frac{p+2}{(p+2)^2+1}\right)$$

$$p^2Y - 3 + 4pY + 5Y = 2\left(\frac{p+2}{(p+2)^2+1}\right)$$

$$Y = \left(2\left(\frac{p+2}{(p+2)^2+1}\right) + 3\right)\cdot\left(\frac{1}{p^2+4p+5}\right)$$

$$Y = 3\left(\frac{1}{(p+2)^2+1}\right) + 2\left(\frac{p+2}{((p+2)^2+1)^2}\right)$$

L14

take a derivative with respect to b in L14

$$\mathcal{L}(-t e^{-at}\sin bt) = \frac{-2b(p+a)}{((p+a)^2+b^2)^2} \quad \text{(A)}$$

$$\mathcal{L}^{-1}(Y) = 3\,\mathcal{L}^{-1}\left(\frac{1}{(p+2)^2+1}\right) - \mathcal{L}^{-1}\left(\frac{-2(p+2)}{((p+2)^2+1)^2}\right)$$

L13 b=1 a=2

(A) b=1 a=2

$$y(t) = 3e^{-2t}\sin t + t e^{-2t}\sin t$$

$$y(t) = (3+t)e^{-2t}\sin t$$

27. Here we want to take Laplace transforms of both y and z and their derivatives. Let Y be the Laplace transform of y as usual, and let Z be the Laplace transform of z. Take Laplace transforms of both differential equations and substitute the initial conditions.

$$\begin{cases}(pY - y_0) + (pZ - z_0) - 3Z = 0, \\ (p^2Y - py_0 - y_0') + (pZ - z_0) = 0,\end{cases}$$

$$\begin{cases}pY + pZ - \frac{4}{3} - 3Z = 0, \\ p^2Y + pZ - \frac{4}{3} = 0.\end{cases}$$

We can solve these equations simultaneously for Y and Z to find

$$Y = \frac{4}{p^2(4 - p)}, \qquad Z = \frac{4}{3}\left(\frac{1}{(p - 4)} - \frac{1}{p(p - 4)}\right).$$

Then, using L2 and L7 to take the inverse transform of Z, we find z:

$$z = \frac{4}{3}\left(e^{4t} - \frac{1 - e^{4t}}{-4}\right) = \frac{4}{3}e^{4t} + \frac{1}{3}(1 - e^{4t})$$
$$= e^{4t} + \frac{1}{3}.$$

We can now either find y from Y, or we can find y' directly from the original differential equations and integrate to get y. By the latter method, we find from the first differential equation

27. (continued)

$$y' = 3z - z' = 3\left(e^{4t} + \frac{1}{3}\right) - 4e^{4t} = 1 - e^{4t},$$
$$y = \int y'\,dt = t - \frac{1}{4}e^{4t} + C.$$

Since $y_0 = 0$, we find $C = 1/4$; then

$$y = t + \frac{1}{4}\left(1 - e^{4t}\right).$$

To find y from Y, write

$$Y = \frac{(4 - p) + p}{p^2(4 - p)} = \frac{1}{p^2} + \frac{1}{p(4 - p)}$$

and use L6 and L7 to get the same y we found above.

29. Take Laplace transforms of both equations and substitute the initial conditions.

$$\begin{cases} pY - 1 + pZ - 1 - 2Y = \frac{1}{p}, \\ Z - (pY - 1) = \frac{1}{p^2}, \end{cases} \qquad \text{or}$$

$$(1) \qquad \begin{cases} (p - 2)Y + pZ = 2 + \frac{1}{p}, \\ -pY + Z = \frac{1}{p^2} - 1. \end{cases}$$

Solving these equations simultaneously, we find

$$Y = \frac{1}{p - 1}$$

so by L2,

$$(2) \qquad y = e^t.$$

29. (continued)

The easiest way to find z is to go back to the given differential equations. Solving the second equation for z and using (2) gives

$$(3) \qquad z = t + y' = t + e^t.$$

Alternatively we could find Z from (1).

$$(p^2 + p - 2)Z = p\left(2 + \frac{1}{p}\right) + (p - 2)\left(\frac{1}{p^2} - 1\right) = p + 2 + \frac{p^2 + p - 2}{p^2},$$

$$Z = \frac{p + 2}{(p + 2)(p - 1)} + \frac{p^2 + p - 2}{p^2(p^2 + p - 2)} = \frac{1}{p - 1} + \frac{1}{p^2}.$$

Then by L2 and L5 we find z as in (3).

15.4

4, 9

let $f(x)$ be even

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\left(\cos \alpha x - i \sin \alpha x\right) dx$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} f(x)\cos \alpha x\, dx - i \underbrace{\int_{-\infty}^{\infty} f(x) \sin \alpha x\, dx}_{0}\right]$$

even $\cdot$ odd = odd

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos \alpha x\, dx$$

even $\cdot$ even = even

$$= \frac{1}{\pi}\int_{0}^{\infty} f(x)\cos \alpha x\, dx$$

so see problems 13, 15

(note: be aware of the constants in both though)

13.)

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(x) \cos \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{\frac{\pi}{2}}^{\pi} \cos \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha} \Big|_{\frac{\pi}{2}}^{\pi} \sin \alpha x$$

$$= \frac{1}{\alpha}\sqrt{\frac{2}{\pi}} \left(\sin \alpha \pi - \sin \alpha \frac{\pi}{2} \right)$$

$$f_c(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha \pi - \sin\left(\alpha \frac{\pi}{2}\right)}{\alpha} \cos \alpha x \, d\alpha$$

15.)

$$S_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^a (-2x + 2a) \cos \alpha x \, dx$$

$$(x \sin \alpha x)' = \sin \alpha x + \alpha x \cos \alpha x$$

$$= \sqrt{\frac{2}{\pi}} \left[2a \int_0^a \cos \alpha x - 2 \int_0^a x \cos \alpha x \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{2a}{\alpha} \Big|_0^a \sin \alpha x - \frac{2}{\alpha} \left(\Big|_0^a x \sin \alpha x - \int_0^a \sin \alpha x \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\cancel{\frac{2a}{\alpha} \sin \alpha a} + \frac{2}{\alpha} \left(\cancel{a \sin \alpha a} - \frac{1}{\alpha} \cos \alpha a + \frac{1}{\alpha} \right) \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos \alpha a}{\alpha^2} \right)$$

$$f_c(x) = \frac{4}{\pi} \int_0^\infty \left(\frac{1 - \cos \alpha a}{\alpha^2} \right) \cos \alpha x \, d\alpha$$

26.) in 15

$a = 1,\ x = 0$

$$\int_0^\infty \left(\frac{1-\cos x}{x^2}\right) dx = \frac{\pi}{4} f_c(0) = \frac{\pi}{\cancel{4}2} \cdot \cancel{2} = \frac{\pi}{2}$$

15.5

4) $\dfrac{1}{(p+a)(p+b)^2}$

$$G(p) = \frac{1}{(p+a)(p+b)} = h\left(\frac{e^{-at} - e^{-bt}}{b-a}\right)$$

$$H(p) = \frac{1}{p+b} = h\left(e^{-bt}\right)$$

$$y = \int_0^t \left(\frac{e^{-a\tau} - e^{-b\tau}}{(b-a)}\right) \cdot e^{-b(t-\tau)}\, d\tau$$

$$= \frac{e^{-bt}}{b-a} \int_0^t \left(e^{-a\tau} e^{b\tau} - e^{-b\tau} e^{b\tau}\right) d\tau$$

$$= \frac{e^{-bt}}{b-a} \left[\int_0^t e^{(b-a)\tau}\, d\tau - \int_0^t 1\, d\tau\right]$$

$$= \frac{e^{-bt}}{b-a} \left[\frac{1}{b-a}\left(e^{(b-a)t} - 1\right) - t\right]$$

15.) $y'' + 3y' - 4y = e^{3t} \quad y_0 = y_0' = 0$

take the laplace transform of each term

$p^2Y - p\cancel{y_0}^{0} - \cancel{y_0'}^{0} + 3(pY - \cancel{y_0}^{0}) - 4Y = L(e^{3t})$

$p^2Y + 3pY - 4Y = L(e^{3t})$

$$Y = \frac{L(e^{3t})}{(p+4)(p-1)} = L(e^{3t})\, L\left(\frac{e^{t} - e^{-4t}}{5}\right)$$

$$Y = \frac{1}{5}\int_0^t (e^{\tau} - e^{-4\tau}) \cdot e^{3t} e^{-3\tau}\, d\tau$$

$$= \frac{e^{3t}}{5}\int_0^t e^{-2\tau} - e^{-7\tau}\, d\tau = \frac{e^{3t}}{5}\left(\Big|_0^t \frac{e^{-2\tau}}{-2} - \frac{e^{-7\tau}}{-7} \right)$$

$$= \frac{e^{3t}}{5}\left(\frac{-e^{-2t}}{2} + \frac{e^{-7t}}{7} + \frac{1}{2} - \frac{1}{7} \right)$$

$$= \boxed{\frac{-e^{t}}{10} + \frac{e^{-4t}}{35} + \frac{e^{3t}}{14}}$$

18) $y'' + w^2 y = f(t) \qquad y = y_0' = 0$

$$L(y'') + w^2 L(y) = L(f(t))$$

$$p^2 Y - p\cancel{y_0}^{\,0} - \cancel{y_0'}^{\,0} + w^2 Y = L(f(t))$$

$$Y = \frac{L(f(t))}{p^2+w^2} = \frac{1}{w} L(f(t)) \left(\frac{w}{p^2+w^2}\right) = \frac{1}{w} L(f(t)) L(\sin wt)$$

$$y = \frac{1}{w}\int_0^t f(\tau)\sin w(t-\tau)\, d\tau$$

$t < a$

$$y = \frac{1}{w}\int_0^t \sin(wt - w\tau)\, d\tau$$

$u = wt - w\tau$

$du = -w\, d\tau$

$$y = \frac{-1}{w^2}\int_{0}^{t} \sin u\, du = \frac{1}{w^2}\Big|_0^t \cos w(t-\tau) = \frac{1}{w^2}(1 - \cos wt)$$

$t > a$

$$y = \frac{1}{w}\int_0^a \sin(wt - w\tau)\, d\tau = \frac{1}{w^2}\Big|_0^a \cos w(t-\tau) = \frac{1}{w^2}(\cos w(t-a) - \cos wt)$$

free vibrations

$$y'' + \omega^2 y = 0$$

$$y'' = -\omega^2 y$$

$$\omega = 2\pi f = \frac{2\pi}{T} \quad \text{or} \quad T = \frac{2\pi}{\omega}$$

$$a = \frac{1}{3}T = \frac{1}{3}\frac{2\pi}{\omega}$$

$$a = \frac{3}{2}T = \frac{3}{\cancel{2}}\frac{\cancel{2}\pi}{\omega}$$

$$a = \frac{1}{10}T = \frac{1}{10}\frac{2\pi}{\omega} = \frac{\pi}{5\omega}$$

let $\omega = \pi$ for simplicity

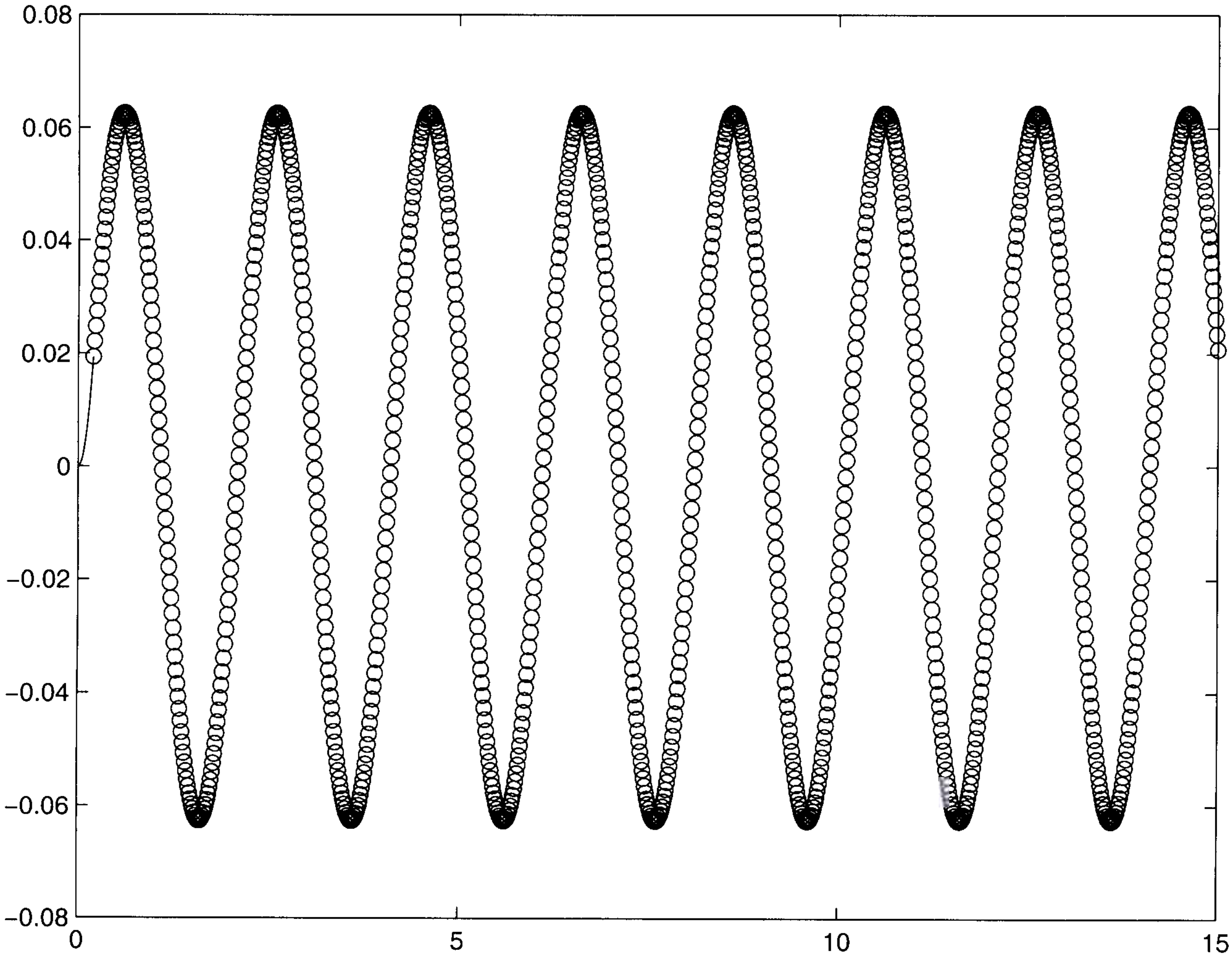
$$a = \frac{2\cancel{\pi}}{3\cancel{\pi}} = \frac{2}{3}$$

$$a = \frac{3\cancel{\pi}}{\cancel{\pi}} = 3$$

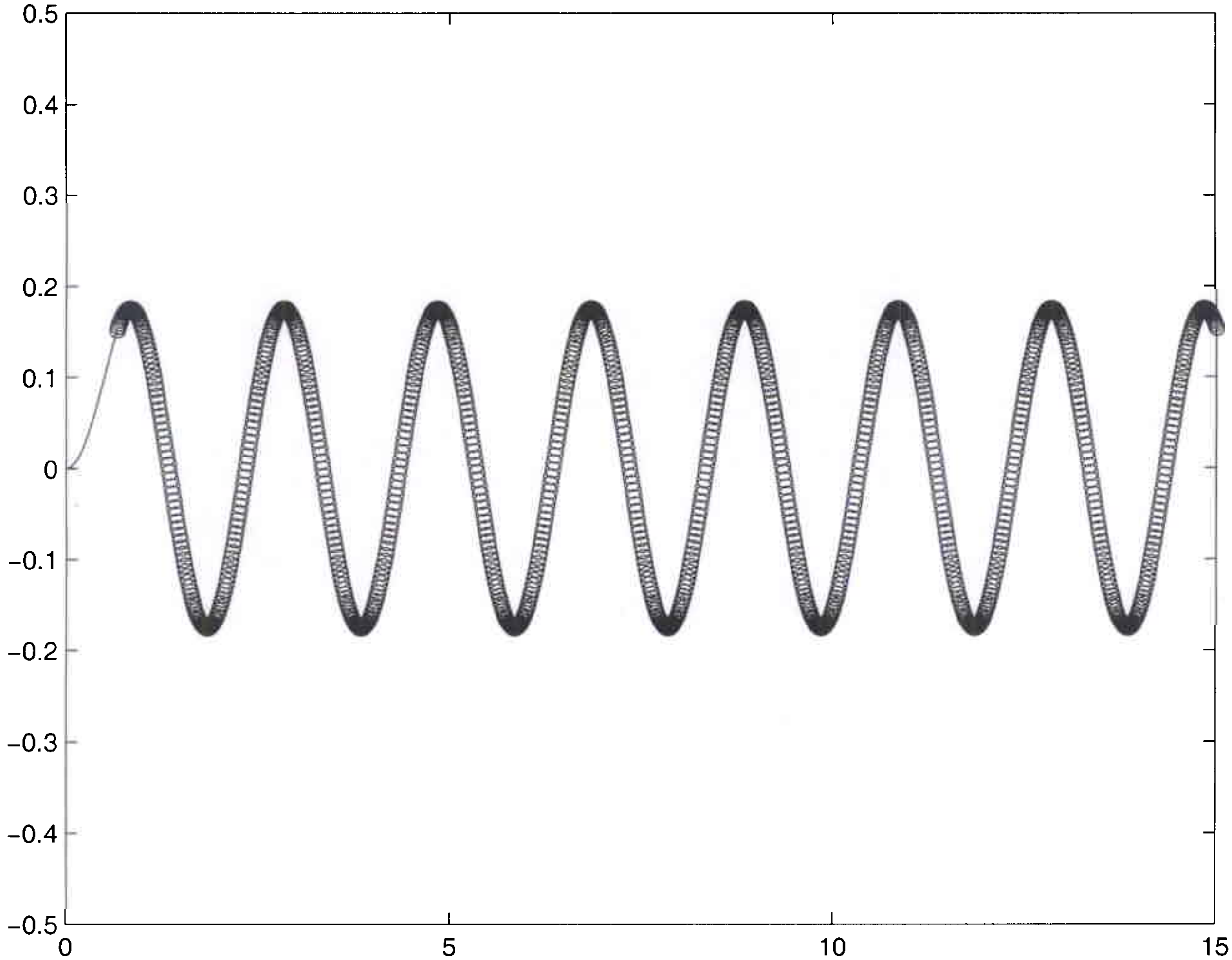
$$a = \frac{\cancel{\pi}}{5\cancel{\pi}} = \frac{1}{5}$$

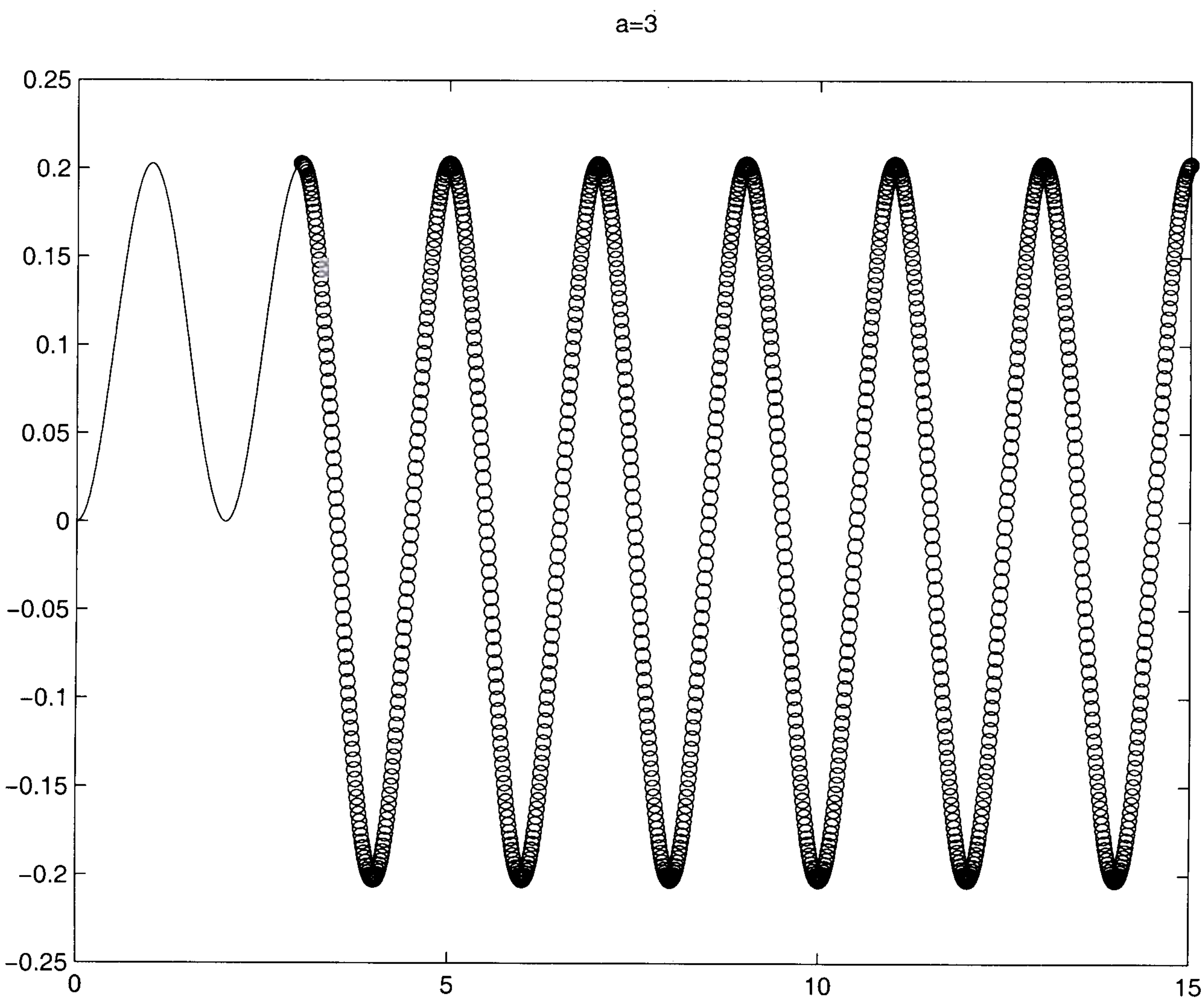
$$y = \begin{cases} \frac{1}{\pi^2}\left(1 - \cos \pi t\right) & t < a \\ \frac{1}{\pi^2}\left(\cos \pi(t-a) - \cos \pi t\right) & t > a \end{cases}$$

a=1/5



a=2/3

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0.15
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0.05
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0
5
10
15