HW 6

1.) 15.7

$$L^{-1}\left(\frac{e^{-2p}}{p^2}\right) = L(t)\,L(\delta(t-2))$$

$$= \int_0^t \tau\,\delta(t-\tau-2)\,d\tau$$

$$= \begin{cases} t-2 & t>2 \\ 0 & \text{o.w.} \end{cases}$$

15.7

5.) $p^2 Y - p\cancel{y_0} - \cancel{y_0'} + \omega^2 Y = L(f_n(t))$

$$Y = \frac{L(f_n(t))}{p^2+\omega^2} = \frac{1}{\omega} L(\sin\omega t) L(f_n(t))$$

for $t > t_0$

there is a N such that when $n \geq N$ $t > t_0 + \frac{1}{n}$ $\forall n$

$$= \frac{1}{\omega} \int_{t_0}^{t_0+\frac{1}{n}} n \sin\omega(t-\tau)\, d\tau \qquad t > t_0,\ n \geq N$$

$$= \frac{n}{\omega^2} \Big|_{t_0}^{t_0+\frac{1}{n}} \cos\omega(t-\tau)$$

$$= \frac{n}{\omega^2}\left[\cos\omega\left((t-t_0)-\frac{1}{n}\right) - \cos\omega(t-t_0)\right]$$

$$= \frac{n}{\omega^2}\left[\cos\omega(t-t_0)\cos\omega\left(\frac{1}{n}\right) + \sin\omega(t-t_0)\sin\omega\left(\frac{1}{n}\right) - \cos\omega(t-t_0)\right]$$

15.1

5) con't

$$= \frac{1}{w^2}\left[\cos w(t-t_0)\left(\frac{\cos w\left(\frac{1}{n}\right)-1}{\frac{1}{n}}\right)+\sin w(t-t_0)\left(\frac{\sin w\left(\frac{1}{n}\right)}{\frac{1}{n}}\right)\right]$$

$$\text{so} \lim_{n\to\infty} y_n(t) = \begin{cases} \frac{\cancel{w}\sin w(t-t_0)}{w^{\cancel{2}}} & t>t_0 \\ = 0 & \text{o.w.} \end{cases}$$

$\underline{\text{2.7}}$

10.1

$$y'' - 9y = \delta(t - t_0) \qquad 1, t \quad y_0 = y_0' = 0$$

$$p^2 Y - p\cancel{y_0} - \cancel{y_0'} - 9Y = L[\delta(t - t_0)]$$

$$Y = \frac{1}{3} L(\delta(t - t_0)) L(\sinh 3t)$$

$$y = \frac{1}{3}\int_0^t \delta(\tau - t_0) \sinh 3(t - \tau)\, d\tau$$

$$y = \begin{cases} \frac{1}{3}\sinh 3(t - t_0) & t > t_0 \\ 0 & t < t_0 \end{cases}$$

6. Hint: Solve by Laplace transforms the differential equation

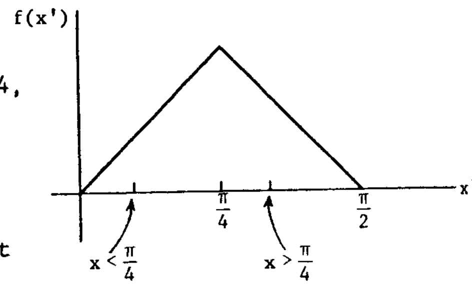
$$G'' - a^2 G = \delta(t - t'), \qquad G_0 = G_0' = 0.$$

13. By text equation (8.17)

$$(1) \qquad y = -(\cos x)\int_0^x (\sin x')f(x')\,dx' - (\sin x)\int_x^{\pi/2} (\cos x')f(x')\,dx'.$$

We sketch the given function

$$(2) \qquad f(x') = \begin{cases} x', & 0 < x' < \pi/4, \\ \frac{\pi}{2} - x', & \frac{\pi}{4} < x' < \frac{\pi}{2}. \end{cases}$$



Now we use the graph to see what $f(x')$ is in each of the integrals in (1). This depends on whether $x < \frac{\pi}{4}$ or $x > \frac{\pi}{4}$.

For $x < \frac{\pi}{4}$:

In the integral from 0 to x, we see from the graph that $f(x') = x'$. But the integral from $x' = x$ to $x' = \frac{\pi}{2}$ must be written in two parts. For x' between x and $\frac{\pi}{4}$, we have $f(x') = x'$, but for x' between $\frac{\pi}{4}$ and $\frac{\pi}{2}$, we have $f(x') = \frac{\pi}{2} - x'$. Thus the integrals in (1) are:

$$\int_0^x (\sin x')f(x')\,dx' = \int_0^x (\sin x')x'\,dx' = \sin x - x\cos x,$$

$$\int_x^{\pi/2} (\cos x')f(x')\,dx' = \int_x^{\pi/4} x'\cos x'\,dx' + \int_{\pi/4}^{\pi/2}\left(\frac{\pi}{2} - x'\right)\cos x'\,dx'$$

$$= \cos x' + x'\sin x'\Big|_x^{\pi/4} + \frac{\pi}{2}\sin x' - (\cos x' + x'\sin x')\Big|_{\pi/4}^{\pi/2}$$

$$= -\cos x - x\sin x + \sqrt{2}$$

13. (continued)

(after some algebra). Substitute these results into (1) to find $y(x)$ when $x < \frac{\pi}{4}$.

$$y(x) = -(\cos x)(\sin x - x\cos x) - (\sin x)(-\cos x - x\sin x + \sqrt{2})$$

$$= x - \sqrt{2}\sin x, \quad x < \frac{\pi}{4}.$$

For $x > \frac{\pi}{4}$:

This time the integral from 0 to x must be written as a sum of two integrals. For x' between 0 and $\pi/4$, we have $f(x') = x'$ (see graph), and for x' between $\pi/4$ and x, we have $f(x') = \frac{\pi}{2} - x'$. For the integral from x to $\pi/2$, we have $f(x') = \frac{\pi}{2} - x'$. Substitute these into the integrals in (1) and evaluate as above to get $y(x)$ for $x > \frac{\pi}{4}$. Thus find

$$y(x) = \begin{cases} x - \sqrt{2}\sin x, & x < \frac{\pi}{4}, \\ \frac{\pi}{2} - x - \sqrt{2}\cos x, & x > \frac{\pi}{4}. \end{cases}$$

It is straightforward to verify that $y'' + y = f(x)$ and that $y(0) = y(\pi/2) = 0$ (check these).

15.8

7.) $$y(t) = \int_0^t \frac{1}{a} \sinh a(t-t') e^{-t'} dt$$

$$= \frac{1}{a} \int_0^t \frac{e^{a(t-t')} - e^{-a(t-t')}}{2} e^{-t'} dt'$$

$$= \frac{1}{2a} \int_0^t e^{at-at'-t'} - e^{-at+at'-t'} dt'$$

$$= \frac{1}{2a} \left[e^{at} \int_0^t e^{t'(-a-1)} - e^{-at} \int_0^t e^{t'(a-1)} \right]$$

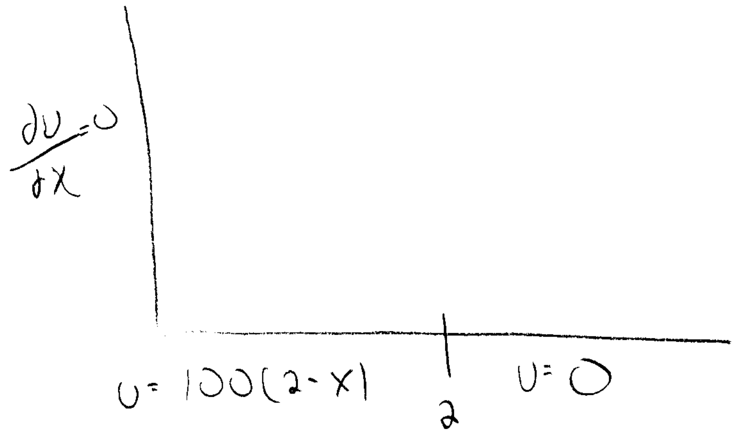
$$= \frac{1}{2a} \left[\frac{e^{at}}{-(a+1)} \left(e^{-t(a+1)} - 1 \right) - \frac{e^{-at}}{a-1} \left(e^{-t(1-a)} - 1 \right) \right]$$

$$= \frac{-1}{2a} \left[\frac{e^{-t} - e^{-at}}{a+1} + \frac{e^{-t} - e^{-at}}{a-1} \right]$$

$$= \frac{e^{-at} - e^{-t}}{a^2-1}$$

15.9

2.)



$u \to 0$ as $y \to \infty$

$\frac{\partial u}{\partial x} = 0$ at $x = 0$

$u = e^{-ky}\cos kx$

$$u(x,y) = \int_0^\infty B(k)\, e^{-ky}\cos kx \, dk$$

$$u(x,0) = \int_0^\infty B(k)\cos kx \, dk$$

$$\sqrt{\frac{\pi}{2}}\, B(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,0)\cos kx \, dx$$

$$= \frac{2}{\pi}\int_0^2 100(2-x)\cos kx \, dx$$

$$= \frac{200}{\pi}\int_0^2 (2-x)\cos kx \, dx$$

15.9

2) cont.

$$= \frac{200}{\pi k}\left[\cancel{\Big|_0^2 (2-x)} \sin kx + \int_0^2 \sin kx \right]$$

$$= \frac{-200}{\pi k^2} \Big|_0^2 \cos kx = \frac{-200}{\pi k^2}\left(\cos 2k - 1 \right)$$

$$u(x,y) = \int_0^\infty \frac{-200}{\pi k^2}\left(\cos 2k - 1 \right) e^{-ky} \cos kx \; dk$$

3. Take the t Laplace transform of the heat flow equation for $u(x,t)$,

(1) $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

with $u_0 = u(x,0) = 100x/\ell$. This gives the differential equation for $U(x,p)$:

3. (continued)

(2) $$\frac{\partial^2 U}{\partial x^2} = \frac{1}{\alpha^2}(pU - u_0) = \frac{1}{\alpha^2}pU - \frac{100x}{\alpha^2\ell}.$$

The solutions of $\frac{\partial^2 U}{\partial x^2} = \frac{p}{\alpha^2}U$ are $\sinh\left(x\sqrt{\frac{p}{\alpha^2}}\right)$ and $\cosh\left(x\sqrt{\frac{p}{\alpha^2}}\right)$.

If $U = Kx$, then $\frac{\partial^2 U}{\partial x^2} = 0$, and $\frac{p}{\alpha^2}U = \frac{100x}{\alpha^2\ell}$ if $K = \frac{100}{p\ell}$.

Thus the general solution of (2) is

(3) $$U(x,p) = A\sinh(xp^{1/2}/\alpha) + B\cosh(xp^{1/2}/\alpha) + \frac{100x}{p\ell}.$$

Since the temperature at $x = 0$ is held at 0° for all t, we have $u(0,t) = 0$ so

$$U(0,p) = [\text{Laplace transform of } u(0,t)] = 0.$$

Similarly

$$u(\ell,t) = 0, \qquad U(\ell,p) = 0.$$

Substitute these values into (3) to get

$$B = 0, \quad A\sinh(\ell p^{1/2}/\alpha) + \frac{100}{p} = 0, \quad A = \frac{-100}{p\sinh(\ell p^{1/2}/\alpha)}.$$

Thus (3) becomes

$$U(x,p) = -\frac{100\sinh(xp^{1/2}/\alpha)}{p\sinh(\ell p^{1/2}/\alpha)} + \frac{100x}{p\ell}.$$

We assume the expansion given:

$$\frac{100\sinh(xp^{1/2}/\alpha)}{p\sinh(\ell p^{1/2}/\alpha)} = \frac{100x}{p\ell} - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi x/\ell)}{n[p + (n\pi\alpha/\ell)^2]}.$$

Take inverse Laplace transforms of each of the terms using L2 (text page 636) to get

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-(n\pi\alpha/\ell)^2 t}\sin(n\pi x/\ell)$$

as on text page 552, equation (3.15).

15.9

5.) $u(0,t) = 2\sin 3t \quad t > 0$

$$u(x,0) = 0 \, , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$$U(x,p) = \int_0^{\infty} u(x,t)\, e^{-pt}\, dt$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{v^2}\left(p^2 U - \cancel{p u_{t=0}} - \cancel{u'_{t=0}}\right)$$

$$\frac{\partial^2 U}{\partial x^2} = \left(\frac{p}{v}\right)^2 U$$

$u = 3\sin 3t \quad$ at $x = 0 \quad \rightarrow \quad U = L(2\sin 3t) = \dfrac{6}{p^2+3^2}$ at $x=0$

$u \rightarrow 0 \quad$ as $x \rightarrow \infty \qquad U \rightarrow L(0) = 0 \quad$ as $x \rightarrow \infty$

15.9

5.) cont.

$$U = \frac{6}{p^2+3^2} e^{-\frac{p}{v}x}$$

$$L^{-1}(U) = 2L(\delta(t-\tfrac{x}{v}))L(\sin 3t)$$

$$u(x,t) = 2\int_0^t \delta(\tau - \tfrac{x}{v}) \sin 3(t-\tau)\, d\tau$$

$$= \begin{cases} 2\sin 3(t-\frac{x}{v}) & t > \frac{x}{v} \\ 0 & \text{o.w.} \end{cases}$$

12.1

4.) $y'' + y = 4\sin 3x$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots$$

$$\sin 3x = 3x - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \dots$$

$$a_0 + 2a_2 = 0 \longrightarrow a_2 = -\frac{1}{2}a_0$$

$$a_2 + 12a_4 = 0 \longrightarrow a_4 = -\frac{1}{12}a_2 = -\frac{1}{12}\cdot -\frac{1}{2}a_0$$

n even

a_0

$a_2 = -\frac{a_0}{2!}$

$a_4 = \frac{a_0}{4!}$

⋮

12.1

4) cont.

$$a_1 + 6a_3 = 12 \longrightarrow a_3 = 2 - \frac{1}{6}a_1$$

$$a_3 + 20a_5 = -18 \longrightarrow a_5 = -\frac{9}{10} - \frac{a_3}{20}$$

$$= -\frac{9}{10} - \frac{1}{20}\left(2 - \frac{1}{6}a_1\right)$$

$$= -1 + \frac{a_1}{120}$$

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 + \left(2 - \frac{1}{6}a_1\right)x^3 + \frac{a_0}{4!}x^4$$

$$+ \left(-1 + \frac{a_1}{120}\right)x^5 + \ldots$$

12.41

4) cont $y'' + y = 4\sin 3x$

particular

$y_p = C\sin 3x$

$-9C\sin 3x + C\sin 3x = 4\sin 3x$

$\rightarrow C = -\frac{1}{2}$

$y_p = -\frac{1}{2}\sin 3x$

homogeneous

$y'' = -y$

$y_h = A\sin x + B\cos x$

fundamental

$y = y_p + y_h = A\sin x + B\cos x - \frac{1}{2}\sin 3x$

12.1

4.) cont.

$$A\sin x = A\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$-\frac{1}{2}\sin 3x = -\frac{1}{2}\left(3x - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \cdots\right)$$

$$B\cos x = B\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$y = B + \left(A - \frac{3}{2}\right)x - \frac{B}{2!}x^2 - \underbrace{\frac{1}{3!}\left(A - \frac{3^3}{2}\right)}_{-\frac{1}{6}\left(\left(A-\frac{3}{2}\right)+\frac{3}{2}-\frac{27}{2}\right)}x^3 + \cdots$$

Let $B = a_0$, $\left(A - \frac{3}{2}\right) = a_1$

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 + \frac{1}{6}\left(12 - a_1\right) + \cdots$$

12.

12.) $y'' = (x^2+1)y$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+2} = a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \dots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = (2\cdot 1)a_2 + (3\cdot 2)a_3 x + (4\cdot 3)a_4 x^2 + \dots$$

$a_0 = (2\cdot 1)a_2$

$a_1 = (3-2)a_3$

$a_2 + a_0 = (4\cdot 3)a_4$

$a_3 + a_1 = (5\cdot 4)a_5$

$a_4 + a_2 = (6\cdot 5)a_6$

$a_5 + a_3 = (7\cdot 6)a_7$

12.1 12) cont.

$$a_2 = \frac{a_0}{2}$$

$$a_4 = \frac{1}{12}\left(a_2 + a_0\right) = \frac{1}{12}\left(\frac{a_0}{2} + a_0\right) = \left(\frac{1}{\cancel{12}4} \cdot \frac{\cancel{3}}{2}\right) a_0 = \frac{a_0}{8}$$

$$a_6 = \frac{1}{30}\left(a_4 + a_2\right) = \frac{1}{30}\left(\frac{a_0}{8} + \frac{a_0}{2}\right) = \frac{1}{\cancel{30}6}\left(\frac{\cancel{5}a_0}{8}\right) = \frac{a_0}{48}$$

$$a_3 = \frac{1}{6} a_1$$

$$a_5 = \frac{1}{20}\left(a_1 + a_3\right) = \frac{1}{20}\left(\frac{1}{6}a_1 + a_1\right) = \frac{1}{20}\left(\frac{7}{6}a_1\right) = \frac{7}{120}a_1$$

$$a_7 = \frac{1}{42}\left(a_5 + a_3\right) = \frac{1}{42}\left(\frac{7}{120}a_1 + \frac{1}{6}a_1\right) = \frac{1}{42}\left(\frac{27}{120}\right)a_1$$

$$= \frac{3}{560}a_1$$

$$y = a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_0}{8}x^4 + \frac{7}{120}a_1 x^5 + \cdots$$

1. Let us find P_2 using text equation (2.7). For $\ell = 2$, we see that the coefficient of x^4 is zero and, by text equation (2.6), all the following coefficients of even powers of x are zero also. Thus the a_0 series is just $1 - 3x^2$. Remember that a_0 and a_1 are arbitrary. We set $a_1 = 0$ here since the a_1 series is an infinite series and we want a polynomial. Then (with $\ell = 2$, $a_1 = 0$)

$$y = a_0(1 - 3x^2).$$

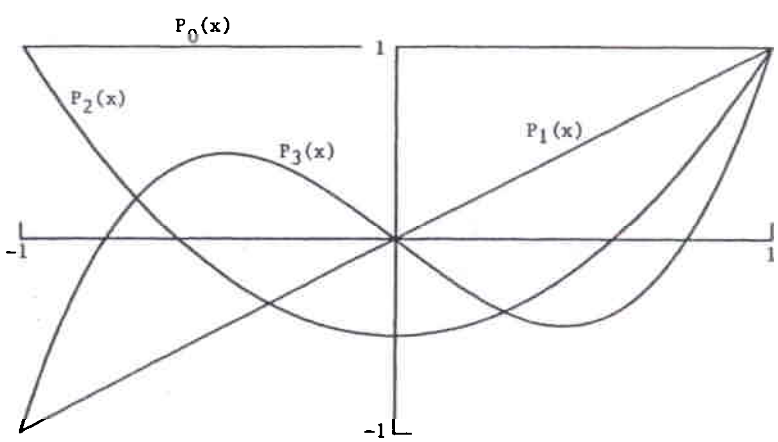
Legendre polynomials are required to be 1 when $x = 1$. We get

$$1 = a_0(1 - 3) \qquad \text{or} \qquad a_0 = -1/2, \qquad \text{so}$$

$$P_2(x) = -\frac{1}{2}(1 - 3x^2) = \frac{1}{2}(3x^2 - 1)$$

as in text equation (2.8). Similarly, to find $P_3(x)$ from text equation (2.7), let $\ell = 3$, $a_0 = 0$, and find a_1 to make $y = 1$ when $x = 1$. To find P_4, let $\ell = 4$, etc. To check your answers, see Problems 4.3 and 5.3 below.

3. Graphs of Legendre polynomials.



Note that the graphs agree with the following facts:

$P_\ell(1) = 1$ for all ℓ.

$P_\ell(0) = 0$ for odd ℓ (but not for even ℓ).

$$P_\ell(-1) = (-1)^\ell = \begin{cases} 1, & \ell \text{ even}, \\ -1, & \ell \text{ odd}. \end{cases}$$

2.1 $P_\ell(x)$

When ℓ is even

$P_\ell(x)$ consists of a sum of only even functions
so it is even $P_\ell(x) = P_\ell(-x)$

When ℓ is odd

$P_\ell(x)$ consists of a sum of only odd functions
so it is odd $P_\ell(x) = -P_\ell(-x)$

and we have $P_\ell(1) = 1$

thus $P_\ell(-1) = \begin{cases} P_\ell(1) = 1 & \ell \text{ even} \\ -P_\ell(1) = -1 & \ell \text{ odd} \end{cases}$

or

$$P_\ell(-1) = (-1)^\ell$$