HW#7

12.4

4.) look at $\int_{-1}^{1} x^m P_\ell(x)$ $\quad m<\ell$

- the Legendre polynomials are a complete set on $(-1,1)$
so x^m can be expanded in terms of them

- using $P_0(x)=1$, $P_1(x)=x$
and $\ell P_\ell(x) = (2\ell-1)x P_{\ell-1}(x) - (\ell-1)P_{\ell-2}(x)$
we note that the highest power of x in $P_\ell(x)$ is x^ℓ

- since the set $\{x^n\}$ $n=0,1,2,\ldots$ is linearly independent
the expansion of x^m cannot contain a $P_\ell(x)$
with $\ell>m$

- lastly using the fact that a complete set is an
orthogonal set we arrive at

$$\int_{-1}^{1} x^m P_\ell(x) = 0 \quad \text{when } m<\ell$$

Section 4

3. To find P_3 from Rodrigues' formula, we set $\ell = 3$ in text equation (4.1).

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2 - 1)^3 = \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{48}(6\cdot 5\cdot 4x^3 - 3\cdot 4\cdot 3\cdot 2x) = \frac{1}{2}(5x^3 - 3x).$$

Similarly find the other Legendre polynomials and check your results with the text answers to Problem 5.3, page 766.

Section 5

3. The easiest way to find P_4 (if we know P_2 and P_3) is to use text equation (5.8a) with $\ell = 4$.

$$4P_4 = 7xP_3 - 3P_2 = 7x\cdot\frac{1}{2}(5x^3 - 3x) - 3\cdot\frac{1}{2}(3x^2 - 1)$$

$$= \frac{1}{2}(35x^4 - 21x^2 - 9x^2 + 3),$$

$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Similarly find P_5 and P_6 using P_3 and P_4; check your results with the text answers on page 766.

14. As in Problem 12, we can write any polynomial of degree n in terms of Legendre polynomials with $\ell \leq n$. We start by writing x^n in terms of $P_n(x)$ and a polynomial of degree $n - 1$ or less. We combine terms and then write the x^{n-1} term (or the highest power remaining) in terms of its P_ℓ and so on until we reach P_1 and P_0.

12.5

9.) $3x^2 - 1 + x$

$A(P_2(x)) + B(P_0(x)) = 3x^2 - 1$

$C(P_1(x)) = X$

$A\left(\frac{3x^2}{2} - \frac{1}{2}\right) + B(1) = 3x^2 - 1$

$CX = X$

$\rightarrow C = 1$

$\frac{\cancel{3}}{2} A = \cancel{3}$ $\quad -\frac{1}{2}A + B = -1$

$A = 2$ $\quad -1 + B = -1$

$B = 0$

So $3x^2 + x - 1 = 2P_2(x) + 1P_1(x)$

13.)

$A(P_5(x)) + B(P_3(x)) + C(P_1(x)) = X^5$

$A\left(\frac{63}{8}X^5 - \frac{70}{8}X^3 + \frac{15}{8}X\right) + B\left(\frac{5}{2}X^3 - \frac{3}{2}X\right) + C(X) = X^5$

$A\frac{63}{8} = 1$ $\quad -\frac{70A}{8} + \frac{5}{2}B = 0$ $\quad \frac{15}{8}A - \frac{3}{2}B + C = 0$

$\rightarrow A = \frac{8}{63}$ $\quad (+)\frac{70}{\cancel{8}} \cdot \frac{\cancel{8}}{63} = (+)\frac{5}{2}B$ $\quad C = \frac{3}{\cancel{2}} \cdot \frac{\overset{14}{\cancel{28}}}{63} - \frac{15}{\cancel{8}} \cdot \frac{\cancel{8}}{63}$

$\rightarrow B = \frac{28}{63}$ $\quad C = \frac{27}{63}$

$$\boxed{X^5 = \frac{1}{63}\left(8P_5(x) + 28P_3(x) + 27P_1(x)\right)}$$

12.6

4.) $\overline{f(x)}\, g(x)$ is odd

$$\int_{-a}^{a} \overline{f(x)}\, g(x)\, dx = 0$$

note: taking the complex conjugate will preserve the symmetry of the function

7.) $$b_n = \int_{-\pi}^{\pi} 1 \sin nx \, dx$$

again 1 is even

$\sin nx$ is odd

thus $1 \cdot \sin nx$ is odd

$\rightarrow b_n = 0 \quad \forall n$

8.) $$\int_0^{\pi} \cos\left(\left(n+\tfrac{1}{2}\right)x\right) \cos\left(\left(m+\tfrac{1}{2}\right)x\right) dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos\left(\left(n+\tfrac{1}{2}\right)x\right) \cos\left(\left(m+\tfrac{1}{2}\right)x\right) dx$$

use the proof for 7.5.2 (p.308)
noting that $\left(m+\frac{1}{2}\right) - \left(n+\frac{1}{2}\right)$ and $\left(m+\frac{1}{2}\right) + \left(n+\frac{1}{2}\right)$ are integers for $m, n \in \mathbb{Z}$

$= 0$ iff $m \neq n$

12.6

8) cont

$$a_n = \int_0^{\pi} 1 \cdot \cos\left(\left(n+\tfrac{1}{2}\right)x\right)\,dx$$

$$= \frac{1}{n+\frac{1}{2}} \Big|_0^{\pi} \sin\left(\left(n+\tfrac{1}{2}\right)x\right)$$

$$= \frac{1}{n+\frac{1}{2}} \sin\left(n+\tfrac{1}{2}\right)\pi = \frac{(-1)^n}{n+\frac{1}{2}}$$