12.7

2.)

$$= \frac{d}{dx}\left[(1-x^2)\,P_\ell'(x)\right] + \left[\ell(\ell+1) - (1-x^2)^{-1}\right]P_\ell(x) = 0$$

$$= P_m(x)\,\frac{d}{dx}\left[(1-x^2)\,P_\ell'(x)\right] - P_\ell(x)\frac{d}{dx}\left[(1-x^2)\,P_m'(x)\right]$$

$$+ P_m(x)\,P_\ell(x)\left[\ell(\ell+1) - \cancel{(1-x^2)^{-1}} - m(m+1) + \cancel{(1-x^2)^{-1}}\right] = 0$$

$$= \frac{d}{dx}\left[(1-x^2)\left(P_m(x)P_\ell'(x) - P_\ell(x)P_m'(x)\right)\right.$$

$$+ P_m(x)P_\ell(x)\left[\ell(\ell+1) - m(m+1)\right] = 0$$

$$= \Big|_{-1}^{1}\ \overset{0}{\cancel{(1-x^2)}}\left(P_m(x)\,P_\ell'(x) - P_\ell(x)\,P_m'(x)\right)$$

$$+ \left[\ell(\ell+1) - m(m+1)\right]\int_{-1}^{1} P_m(x)\,P_\ell(x)\,dx = 0$$

12.7

2.) cont.

$$= \left[\ell(\ell+1) - n(n+1) \right] \int_{-1}^{1} P_m(x) P_\ell(x)\, dx = 0$$

iff $m \neq \ell$ $\int_{-1}^{1} P_m(x) P_\ell(x)\, dx = 0$

5.) using 7.1

$\int_{-1}^{1} P_\ell(x) P_0(x) = 0$ when $\ell \neq 0$

and $P_0(x) = 1$

thus $\int_{-1}^{1} P_\ell(x) \cdot 1 = 0$ when $\ell \neq 0$

12.8

4.) $e^{\frac{-x^2}{2}}$ $(-\infty,\infty)$

$$\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} e^{\frac{-x^2}{2}}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx = N^2$$

let $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

look at $I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta = 2\pi\int_0^{\infty} e^{-r^2}\,r\,dr$$

$$= \cancel{2}\pi \cdot \frac{-1}{\cancel{2}}\int_0^{-\infty} e^{u}\,du = -\pi(-1) = \pi$$

thus $I^2 = \pi$

so $I = \sqrt{\pi}$

so $N = \pi^{\frac{1}{4}}$ and $f(x) = \frac{1}{\pi^{\frac{1}{4}}} e^{\frac{-x^2}{2}}$

12.9

5.) $f(x) = \sum_{l=0}^{\infty} C_l P_l(x)$

$$C_l = \left(\frac{2l+1}{2}\right) \int_{-1}^{1} f(x) P_l(x)$$

since $f(x)$ is an even function

$C_l = 0$ when l is odd

$$C_0 = \left(\frac{2\cdot 0+1}{2}\right) \int_{-1}^{1} f(x)\cdot 1\, dx = \frac{1}{\cancel{2}}\cdot \cancel{2} \int_{0}^{1} (-x+1)\, dx$$

$$= \Big|_{0}^{1} -\frac{x^2}{2} + x = \frac{1}{2}$$

$$C_2 = \left(\frac{2\cdot 2+1}{2}\right)\cdot \cancel{2} \int_{0}^{1} (-x+1)\cdot \frac{1}{\cancel{2}}(3x^2-1) = \frac{5}{2}\int_{0}^{1} (-3x^3+3x^2+x-1)\, dx$$

$$= \frac{5}{2} \Big|_{0}^{1} -3\frac{x^4}{4} + \cancel{3}\frac{x^3}{\cancel{3}} + \frac{x^2}{2} - x = \frac{5}{2}\left(\frac{-3}{4} + \cancel{\frac{4}{4}} + \frac{2}{4} - \cancel{\frac{4}{4}}\right)$$

$$= \frac{5}{2}\cdot\frac{-1}{4} = \frac{-5}{8}$$

$$f(x) = \frac{1}{2} P_0(x) - \frac{5}{8} P_2(x) + \dots$$

12.9

16.)

$$f(x) = \sum_{l=0}^{\infty} c_l p_l(x)$$

$$\int_{-1}^{1} p_m(x) f(x)\, dx = \sum_{l=0}^{\infty} c_l \int_{-1}^{1} p_l(x) p_m(x)\, dx$$

use the fact that $\int_{-1}^{1} p_l(x) p_m(x)\, dx = \delta_{lm}$

$$\int_{-1}^{1} p_m(x) f(x)\, dx = c_m$$

so up to $l=2$

$f(x) = c_0 p_0(x) + c_1 p_1(x) + c_2 p_2(x)$ with $c_l = \int_{-1}^{1} p_l(x) f(x)\, dx$

12.9
16.) cont.

$$I = \int_{-1}^{1} \left[f - (b_0 p_0 + b_1 p_1 + b_2 p_2) \right]^2 dx$$

$$= \int_{-1}^{1} \Big(f^2 - 2b_0 p_0 f - 2b_1 p_1 f - 2b_2 p_2 f$$

$$+ b_0^2 p_0^2 + b_1^2 p_1^2 + b_2^2 p_2^2$$

$$+ 2b_0 b_1 \cancel{p_0 p_1}^{\,0} + 2b_0 b_2 \cancel{p_0 p_2}^{\,0} + 2b_1 b_2 \cancel{p_1 p_2}^{\,0} \Big) dx$$

- Use the fact that $\int_{-1}^{1} p_\ell p_m \, dx = \delta_{\ell m}$

- and $c_\ell = \int_{-1}^{1} f p_\ell \, dx$

$$I = \int f^2 dx - 2\left(b_0 c_0 + b_1 c_1 + b_2 c_2\right) + b_0^2 + b_1^2 + b_2^2$$

$$+ \left(c_0^2 + c_1^2 + c_1^2\right) - \left(c_0^2 - c_1^2 - c_2^2\right)$$

12.9
14) cont.

$$I = \int f^2\,dx + (b_0 - c_0)^2 + (b_1 - c_1)^2 + (b_2 - c_2)^2$$

$$- (c_0^2 - c_1^2 - c_2^2)$$

to make I as small as possible with respect to b_0, b_1, b_2

we need only to look at $(b_0 - c_0)^2 + (b_1 - c_1)^2 + (b_2 - c_2)^2$

this is always ≥ 0

the smallest it can be is 0 and this occurs when

$b_0 = c_0$

$b_1 = c_1$

$b_2 = c_2$

this can be easily be done for higher polynomials also

with $f(x) = \sum_{l=0}^{n} c_l P_l(x)$, $c_l = \int_{-1}^{1} f(x) P_l(x)\,dx$, and

the polynomial of degree $n = \sum_{l=0}^{n} b_l P_l(x)$

12.10

1) $(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0$

$y = (1-x^2)^{\frac{m}{2}}u$

$y' = (1-x^2)^{\frac{m}{2}}u' + \frac{m}{\cancel{2}}u(1-x^2)^{\frac{m}{2}-1}\cdot -\cancel{2}x$

$y'' = (1-x^2)^{\frac{m}{2}}u'' + \frac{m}{\cancel{2}}(1-x^2)^{\frac{m}{2}-1}u'\cdot(-\cancel{2}x) + \frac{m}{\cancel{2}}u'(-\cancel{2}x)(1-x^2)^{\frac{m}{2}-1}$

$-\cancel{2}\frac{m}{\cancel{2}}u(1-x^2)^{\frac{m}{2}-1} - (\cancel{2}x)\frac{m}{\cancel{2}}u\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}\cdot(-2x)$

$(1-x^2)^{\frac{m}{2}+1}u'' - mu'x(1-x^2)^{\frac{m}{2}} - mu'x(1-x^2)^{\frac{m}{2}}$

$-mu(1-x^2)^{\frac{m}{2}} + 2x^2u\,m\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-1}$

$-2xu'(1-x^2)^{\frac{m}{2}} + 2x^2mu(1-x^2)^{\frac{m}{2}-1}$

$+\left[l(l+1) - \frac{m^2}{1-x^2}\right](1-x^2)^{\frac{m}{2}}u = 0$

12.10

1. cont.

$$(1-x^2)^{\frac{m}{2}+1} u'' - 2mu'x(1-x^2)^{\frac{m}{2}} - 2x(1-x^2)^{\frac{m}{2}} u'$$

$$- mu(1-x^2)^{\frac{m}{2}} + \cancel{2}x^2 u\left(\frac{m^2 - \cancel{2m}}{\cancel{x}}\right)(1-x^2)^{\frac{m}{2}-1}$$

$$\cancel{+2x^2 um(1-x^2)^{\frac{m}{2}-1}} + l(l+1)(1-x^2)^{\frac{m}{2}} u$$

$$- m^2(1-x^2)^{\frac{m}{2}-1} u = 0$$

$$(1-x^2)u'' - 2xu'(m+1) - mu + x^2 u m^2 (1-x^2)^{-1}$$

$$+ l(l+1)u - m^2(1-x^2)^{-1} u = 0$$

$$\boxed{(1-x^2)u'' - 2(m+1)xu' + \left[l(l+1) - m(m+1)\right]u = 0}$$

12.10

1. cont.

$$\left((1-x^2)u''\right)' = (1-x^2)'u'' + (1-x^2)u'''$$

$$= -2x\,\underline{u'' + (1-x^2)}\,u'''$$

$$-2(m+1)(xu')' = \underline{-2(m+1)u' - 2(m+1)xu''}$$

so

$$\boxed{(1-x^2)u''' - 2(m+2)xu'' + \left[l(l+1)-(m+1)(m+2)\right]u' = 0}$$

Section 10

2. If $x = \cos\theta$, then by the chain rule (text, Chapter 4, Section 5)

$$(1) \qquad \frac{df}{d\theta} = \frac{df}{dx}\frac{dx}{d\theta} = -\sin\theta\,\frac{df}{dx} \qquad \text{or} \qquad \frac{1}{\sin\theta}\frac{df}{d\theta} = -\frac{df}{dx}\,.$$

Write the first term of the given differential equation as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[(\sin^2\theta)\,\frac{1}{\sin\theta}\frac{dy}{d\theta}\right].$$

Then

$$\frac{1}{\sin\theta}\frac{dy}{d\theta} = -\frac{dy}{dx} \qquad \text{and} \qquad \frac{1}{\sin\theta}\frac{d}{d\theta}[\qquad] = -\frac{d}{dx}[\qquad].$$

Also replace $\sin^2\theta$ by $1 - \cos^2\theta = 1 - x^2$. Then

$$\begin{aligned}\frac{1}{\sin\theta}\frac{d}{d\theta}\left[(\sin^2\theta)\,\frac{1}{\sin\theta}\frac{dy}{d\theta}\right] &= -\frac{d}{dx}\left[(1-x^2)\left(-\frac{dy}{dx}\right)\right]\\ &= \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]\\ &= (1-x^2)y'' - 2xy'.\end{aligned}$$

These are the first two terms of text equation (10.1). The rest of the terms are the same as text equation (10.1) if we replace $\sin^2\theta$ by $(1 - x^2)$ as above.

7.1 $D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} \ldots$

$D^{l+m} \ldots$

$= \binom{l+m}{0} D^0 (x-1)^l \, D^{l+m} (x+1)^l$

$+ \binom{l+m}{1} D^1 (x-1)^l \, D^{l+m-1} (x+1)^l$

$+$

$\binom{l+m}{m} D^m (x-1)^l \, D^l (x+1)^l$

$+ \cdots +$

$\binom{l+m}{l} D^l (x-1)^l \, D^m (x+1)^l$

$+$

$\binom{l+m}{l+m} \ldots$

7) cont.

$$= \binom{l+m}{m} D^m (x-1)^l \, D^l (x-1)^l$$

$$+$$

$$\vdots$$

$$+$$

$$\binom{l+m}{l} D^l (x-1)^l \, D^m (x-1)^l$$

$$= \frac{(l+m)!}{m!\, l!} \cdot \frac{l!}{(l-m)!} (x+1)^{l-m} \, l!$$

$$+$$

$$\frac{(l+m)!}{(m+1)!\,(l-1)!} \cdot \frac{l!}{(l-m-1)!} (x+1)^{l-m} \, \frac{l!}{1!} (x-1)^1$$

$$+$$

$$+$$

$$\frac{(l+m)!}{l!\, m!} \cdot l! \, \frac{l!}{(l-m)!} (x-1)^{l-m}$$

12.10

$= $

$+$

$+$

$\vdots$

$= -\left(\right.$

$-\left(\gamma^2-1\right)^{\ell}$

12.10

f.)

$$P_\ell^{-m}(x) = \frac{1}{2^\ell \ell!}(1-x^2)^{-\frac{m}{2}}\frac{d^{\ell-m}}{dx^{\ell-m}}(x^2-1)^\ell$$

$$= \frac{1}{2^\ell \ell!}(1-x^2)^{-\frac{m}{2}}\frac{(\ell-m)!}{(\ell+m)!}(x^2-1)^m\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

$$= \frac{(-1)^m}{2^\ell \ell!}\frac{(\ell-m)!}{(\ell+m)!}(1-x^2)^{\frac{m}{2}}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

$$= (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)$$

Section 11

4. Substitute text equations (11.3) into $x^2y'' - 6y = 0$ and tabulate powers of x as follows:

	x^s	x^{s+1}	$\cdots$	x^{n+s}
x^2y''	$s(s-1)a_0$	$(s+1)sa_1$		$(n+s)(n+s-1)a_n$
$-6y$	$-6a_0$	$-6a_1$		$-6a_n$

Each column (coefficients of a power of x) must add to zero. Thus the indicial equation (first column) gives

$$s^2 - s - 6 = 0, \qquad (s-3)(s+2) = 0, \qquad s = 3 \text{ or } s = -2.$$

For $s = 3$, we find from the second column, $a_1 = 0$. Similarly, from the x^{n+s} column, we find

$$[(n+3)(n+2) - 6]a_n = (n^2 + 5n)a_n = 0$$

so $a_n = 0$ for all $n \neq 0$. The solution corresponding to $s = 3$ is just $a_0x^s = a_0x^3$, or

$$y = Ax^3.$$

For $s = -2$, we also find $a_n = 0$, $n \neq 0$ (verify this), so the corresponding solution is $y = Bx^{-2}$, and the general solution of the given differential equation is

$$y = Ax^3 + Bx^{-2}.$$

(Note that the given differential equation is an Euler or Cauchy equation and so can also be solved as on page 378 of the text.)

14. Hint: Look at the second column of your tabulation of coefficients. What is b_1? Compare the corresponding equation for a_1.