Problem 1 (25 pts) Consider the following heat equation on a 1D rod:

$$u_t = \alpha^2 u_{xx} \quad t \geq 0,\ 0 \leq x \leq \ell,$$

with the boundary condition

$$u(0,t) = u(\ell,t) = 0 \quad \text{for all } t > 0,$$

and the initial condition

$$u(x,0) = Ax(\ell - x),$$

where A is some positive constant.

(a) (6 pts) Use the method of separation of variables to derive the ordinary differential equations (ODEs) with respect to the variable x and t. Then, solve these ODEs to obtain the basic solutions satisfying the above heat equation (not the initial and boundary conditions).

Let $u(x,t) = X(x) \cdot T(t)$

Plug this in $u_t = \alpha^2 u_{xx}$

$\Rightarrow X\dot{T} = \alpha^2 X'' T$

$\Rightarrow \dfrac{X''}{X} = \dfrac{1}{\alpha^2}\dfrac{\dot{T}}{T} = -k^2$, $k \geq 0$ is the separation constant.

Thus we get

$$\begin{cases} X'' + k^2 X = 0 \\ \dot{T} + k^2\alpha^2 T = 0 \end{cases} //$$

So the solutions of these ODEs (with the simplest constants) are

$$X(x) = \begin{cases} \sin kx \\ \cos kx. \end{cases}$$

$$T(t) = e^{-k^2\alpha^2 t} ///$$

(b) (7 pts) Select the basic solutions satisfying the boundary condition, determine the separation constant, and then write a solution to the heat equation by the linear combination of these basic solutions.

Because of the boundary condition

$u(0,t) = u(l,t) = 0$ for all $t > 0$,

we must have $X(x) = \sin kx$ and $\sin kl = 0$

So $k = \frac{n\pi}{l}$, $n = 1, 2, 3, \cdots$

$\therefore$ $X(x) = \sin \frac{n\pi}{l}$, $T(t) = e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}$

Therefore the solution of the heat equation can be written as the linear combination of these functions as

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin\left(\frac{n\pi}{l}x\right)$$

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(c) (7 pts) Finally, write the solution that satifies the initial condition.

Help (rather than hint): You can use the following formula:

$$\int_0^\ell x(\ell - x)\sin\left(\frac{\pi n x}{\ell}\right) dx = 2\left(\frac{\ell}{n\pi}\right)^3 \{1 - (-1)^n\}, \quad n = 1, 2, \ldots.$$

Finally, we need to match the solution with the initial condition at $t=0$.

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right) = Ax(\ell - x).$$

Therefore we need to expand $Ax(\ell-x)$ into the Fourier sine series.

$$b_n = \frac{2}{2\ell}\int_{-\ell}^{\ell} Ax(\ell-x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

$$= \frac{2}{2\ell}\cdot 2A\int_0^{\ell} x(\ell-x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

$$= \frac{2A}{\ell}\cdot 2\left(\frac{\ell}{n\pi}\right)^3\{1-(-1)^n\} \quad \text{by hint} \quad n=1,2,\cdots$$

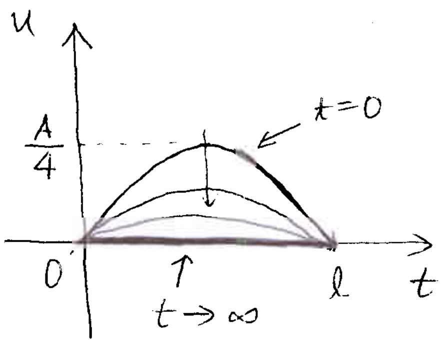
$$= \frac{4A}{\ell}\left(\frac{\ell}{n\pi}\right)^3\{1-(-1)^n\}$$

$$= \frac{4A\ell^2}{(n\pi)^3}\{1-(-1)^n\}$$

$$\therefore u(x,t) = \frac{4A\ell^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3} e^{-\left(\frac{n\pi\alpha}{\ell}\right)^2 t}\sin\left(\frac{n\pi}{\ell}x\right) \quad //$$

$$\text{or} = \frac{8A\ell^2}{\pi^2}\sum_{n:\text{odd}}\frac{1}{n^3} e^{-\left(\frac{n\pi\alpha}{\ell}\right)^2 t}\sin\left(\frac{n\pi}{\ell}x\right) \quad ///$$

(d) (5 pts) Sketch the heat distribution at several time points to indicate how the heat distribution behaves as time progresses.



as $t \to \infty$ $u(x,t) \to 0$ for $0 \le x \le l$.

Problem 2 (25 pts) Consider the following wave equation on a guitar string:

$$u_{tt} = c^2 u_{xx} \quad t \geq 0,\ 0 \leq x \leq \ell,$$

with the boundary condition

$$u(0,t) = u(\ell,t) = 0 \quad \text{for all } t > 0,$$

and the initial condition

$$u(x,0) = A \sin\left(\frac{2\pi x}{\ell}\right),$$

where A is some positive constant.

(a) (20 pts) Solve this wave equation in a similar manner as Problem 1, i.e., describe each step toward the final solution and show your computations.

Use the separation of variables.

Let $u(x,t) = X(x)T(t)$.

Plug it into $u_{tt} = c^2 u_{xx}$.

$\Rightarrow X\ddot{T} = c^2 X'' T$

$\Rightarrow \dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{\ddot{T}}{T} = -k^2$, $k \geq 0$ is a separation constant.

$\Rightarrow \begin{cases} X'' + k^2 X = 0 \\ \ddot{T} + c^2 k^2 T = 0 \end{cases} \Rightarrow X(x) = \begin{cases} \sin kx \\ \cos kx \end{cases} \quad T(t) = \begin{cases} \sin ckt \\ \cos ckt \end{cases}$

Due to the boundary condition $u(0,t) = u(\ell,t) = 0$,

$X(x)$ must be $\sin kx$ with $k = \frac{n\pi}{\ell}$ $n = 1, 2, \ldots$

And the initial condition is not always 0, so

$T(t)$ must be $\cos ckt$,

Thus $X(x)T(t) = \sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi}{\ell}ct\right)$.

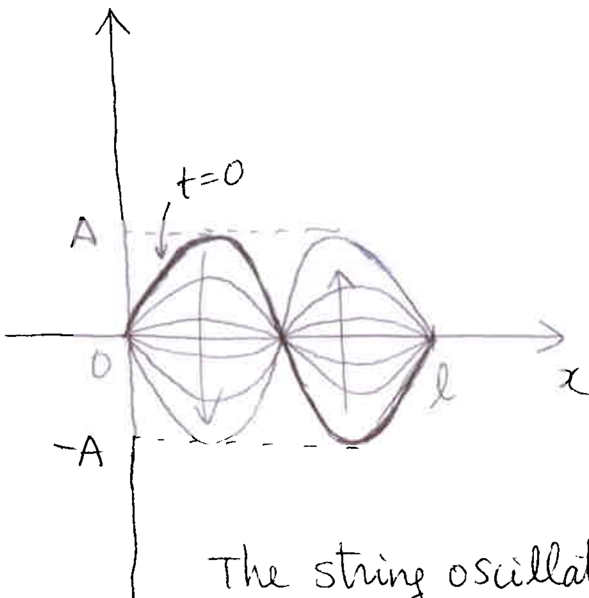
So, $u(x,t) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{\ell}ct\right)\sin\left(\frac{n\pi}{\ell}x\right)$

By matching it with the initial condition at $t=0$

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right) = A\sin\left(\frac{2\pi x}{\ell}\right) \Rightarrow b_n = \begin{cases} A & \text{if } n=2 \\ 0 & \text{otherwise.} \end{cases}$

$\therefore u(x,t) = A\cos\left(\frac{2\pi}{\ell}ct\right)\sin\left(\frac{2\pi}{\ell}x\right)$ ///

(b) (5 pts) Sketch the string behavior at several time points to indicate how the guitar string behaves as time progresses.



The string oscillates indefinitely.

Problem 3 (25 pts) Consider the following *boundary value problem* of the 2nd order differential equation:

$$y'' - 3y' + 2y = 0, \quad y(0) = 0,\ y(1) = 1.$$

(a) (20 pts) Assume that $y'(0) = c$ and solve the above ODE using the Laplace transform. In other words, solve the initial value problem:

$$y'' - 3y' + 2y = 0, \quad y(0) = 0,\ y'(0) = c.$$

$$\mathcal{L}[y'] = pY(p) - y(0) = pY$$

$$\mathcal{L}[y''] = p^2Y(p) - py(0) - y'(0) = p^2Y(p) - c$$

$$\therefore \mathcal{L}[y'' - 3y' + 2] = (p^2 - 3p + 2)Y(p) - c = 0$$

$$\therefore Y(p) = \frac{c}{p^2 - 3p + 2} = \frac{c}{(p-1)(p-2)} = \frac{c}{p-2} - \frac{c}{p-1}$$

$$\therefore y(t) = c(e^{2t} - e^t)$$

(b) (5 pts) Compute the value of c using $y(1) = 1$, and obtain the final solution to the original boundary value problem.

$$y(1) = c(e^2 - e) = 1$$

$$\therefore c = \frac{1}{e^2 - e}$$

$$\therefore y(t) = \frac{e^{2t} - e^t}{e^2 - e}$$

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Problem 4 (25 pts) Let us use the following definition of the Fourier transform:

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}x, \quad x, \xi \in \mathbb{R}.$$

We want to compute the Fourier transform of the following function:

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b]; \\ 0 & \text{otherwise.} \end{cases}$$

In order to do so, let us proceed as follows:

(a) (10 pts) Compute the Fourier transform of

$$\chi_{[-\ell/2,\ell/2]}(x) = \begin{cases} 1 & \text{if } x \in [-\ell/2,\ell/2]; \\ 0 & \text{otherwise.} \end{cases}$$

Then, plug in $\ell = b - a$. This gives us the Fourier transform of the function $\chi_{[-\frac{b-a}{2},\frac{b-a}{2}]}(x)$.

$$\hat{\chi}_{[-\ell/2,\ell/2]}(\xi) = \int_{-\ell/2}^{\ell/2} 1 \cdot e^{-i\xi x}\,dx.$$

$$= \frac{e^{-i\xi\ell/2} - e^{+i\xi\ell/2}}{-i\xi}$$

$$= \frac{-2i\sin(\xi\ell/2)}{-i\xi}$$

$$= \frac{2\sin(\ell\xi/2)}{\xi}$$

$$= \frac{\sin(\ell\xi/2)}{\xi/2}$$

Plug in $\ell = b-a$. to get

$$\hat{\chi}_{[-\frac{b-a}{2},\frac{b-a}{2}]}(\xi) = \frac{\sin(\frac{b-a}{2}\xi)}{\xi/2}$$

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(b) (15 pts) Apply the shift/translation formula of the Fourier transform to obtain the Fourier transform of $\chi_{[a,b]}(x)$.

Hint: The amount of shift is clearly: $a + (b-a)/2 = (a+b)/2$.

$$\mathcal{F}[\tau_\alpha f] = \mathcal{F}[f(x-\alpha)] = e^{-i\xi\alpha}\hat{f}(\xi)$$

$$\therefore \mathcal{F}[\chi_{[a,b]}(x)] = \mathcal{F}\left[\tau_{\frac{a+b}{2}}\chi_{[-\frac{b-a}{2},\frac{b-a}{2}]}(x)\right]$$

$$= e^{-i\frac{a+b}{2}\xi}\,\frac{\sin\left(\frac{b-a}{2}\xi\right)}{\xi/2}$$

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Bonus Problem (20 pts) Consider the following 2D Laplace equation on the rectangular domain,

$$\nabla^2 u = 0 \quad \text{on} \quad \Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le a, 0 \le y \le b\},$$

with the boundary condition,

$$u(x,0) = A\sin\left(\frac{\pi x}{a}\right), u(x,b) = u(0,y) = u(a,y) = 0, \quad 0 \le x \le a, 0 \le y \le b,$$

where A is some positive constant. Solve this Laplace equation in a similar manner as Problems 1 and 2, i.e., describe each step toward the final solution and show your computations.

Huge Hint: $Y(y)$ should be of the form $\sinh k(b-y)$.

Let $u(x,y) = X(x)Y(y)$.

Then $\nabla^2 u = u_{xx} + u_{yy} = X''Y + XY'' = 0$.

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -k^2, \quad k \geq 0 \text{ is a separation constant.}$$

$$\Rightarrow \begin{cases} X'' + k^2 X = 0 \\ Y'' - k^2 Y = 0 \end{cases} \Rightarrow \begin{cases} X(x) = \begin{cases} \sin kx \\ \cos kx \end{cases} \\ Y(y) = \begin{cases} e^{-ky} \\ e^{ky} \end{cases} \end{cases}$$

Because of the boundary condition $u(0,y) = u(a,y) = 0$ we must have $X(x) = \sin kx$ with $\sin ka = 0$

i.e. $k = \frac{n\pi}{a}$, $n = 1, 2, \cdots$

On the other hand the boundary condition $u(x,b) = 0$ gives us $Y(y) = \sinh k(b-y) = \frac{1}{2}\left(e^{k(b-y)} - e^{-k(b-y)}\right)$ by the hint. So $Y(y) = \sinh\left(\frac{n\pi}{a}(b-y)\right)$.

Therefore the solution should be of the form:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right)$$

Set $y=0$ here to get

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi b}{a}\right) = A\sin\left(\frac{\pi x}{a}\right)$$

$$\Rightarrow b_n = \begin{cases} \dfrac{A}{\sinh\left(\frac{\pi b}{a}\right)} & \text{if } n=1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{So, } u(x,y) = \frac{A}{\sinh\left(\frac{\pi b}{a}\right)} \cdot \sin\left(\frac{\pi x}{a}\right) \sinh\left(\frac{\pi(b-y)}{a}\right)$$

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