21.1. (a) $\int_{-\infty}^{\infty} \frac{|\sin x|}{|x|^{\frac{3}{2}}}\, dx = 2\int_0^\infty \frac{|\sin x|}{|x|^{\frac{3}{2}}}\, dx = 2\int_0^1 \frac{|\sin x|}{|x|^{\frac{3}{2}}}\, dx + 2\int_1^\infty \frac{|\sin x|}{|x|^{\frac{3}{2}}}\, dx$

$\le 2\cdot\int_0^1 \frac{|x|}{|x|^{\frac{3}{2}}}\, dx + 2\int_1^\infty \frac{1}{|x|^{\frac{3}{2}}}\, dx. < \infty$

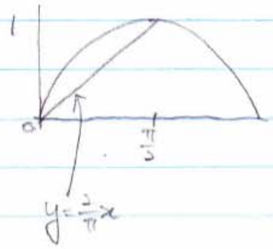
$(\because \int_1^\infty \frac{1}{x^p}\, dx < \infty$ if $p > 1)$

$\int_{-\infty}^\infty \left(\frac{|\sin x|}{|x|^{\frac{3}{2}}}\right)^2 \cdot dx = \int_{-\infty}^\infty \frac{|\sin^2 x|}{|x|^3}\, dx = 2\int_0^\infty \frac{|\sin^2 x|}{|x|^3} = 2\int_0^1 \frac{|\sin^2 x|}{|x|^3}\, dx + 2\int_1^\infty \frac{|\sin^2 x|}{|x|^3}\, dx$

$\int_0^1 \frac{|\sin^2 x|}{|x|^3}\, dx \ge \int_0^1 \frac{(\frac{2}{\pi})^2 x^2}{|x|^3}\, dx$ $(\because \sin x \ge x\cdot\frac{2}{\pi}$ $0 \le x \le \frac{\pi}{2}$.

$= \int_0^1 (\frac{2}{\pi})^2 \cdot \frac{1}{x}\, dx$

$\to \infty.$



$\therefore$ Not in L^2.

(2) $\int_{-\infty}^\infty \left|\frac{1}{(1+x^2)^{1/2}}\right| dx = 2\int_0^\infty \frac{1}{(1+x^2)^{\frac{1}{2}}}\, dx$ $x = \sinh\theta$ $dx = \cosh\theta\cdot d\theta$

$= 2\int_0^\infty \frac{1}{\cosh\theta}\cdot\cosh\theta\cdot d\theta \Rightarrow \infty$ $\notin L^1$

$\int_{-\infty}^\infty \left|\frac{1}{(1+x^2)^{\frac{1}{2}}}\right|^2 \cdot dx = 2\int_0^\infty \frac{1}{1+x^2}\, dx = 2\cdot\tan^{-1}x\Big|_0^\infty = \pi$ $\in L^2$.

(3) $\int_{-\infty}^\infty \left|\frac{1}{x^2-1}\right| \cdot dx = 2\int_0^\infty \frac{1}{|x^2-1|}\, dx = 2\int_0^1 \frac{1}{|x^2-1|}\, dx + 2\int_1^\infty \frac{1}{|x^2-1|}\, dx$

$\int_0^1 \frac{1}{|x^2-1|}\, dx = \int_0^1 \frac{1}{1-x^2}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta}\, d\theta$ by $x = \sin\theta$.
$= \int \sec\theta\cdot d\theta$.

$\Rightarrow \infty$,

$\int_{-\infty}^\infty \frac{1}{|x^2-1|^2}\cdot dx = 2\int_0^\infty \frac{1}{(x^2-1)^2}\, dx > 2\int_0^1 \frac{1}{(x^2-1)^2}\, dx + 2\int_1^\infty \frac{1}{(x^2-1)^2}\, dx$

$2\int_0^1 \frac{1}{(x^2-1)^2}\, dx > 2\int_0^1 \frac{1}{1-x^2}\, dx$. $(\because (x^2-1)^2 < 1-x^2$ for $0 < x < 1$.)

Since $\int_0^1 \frac{1}{1-x^2}\, dx$ diverges, $\int_0^1 \frac{1}{(x^2-1)^2}\, dx$ diverges.

(4). $\frac{1-\cos x}{x^2}$ $\qquad \int_{-\infty}^{\infty} \left|\frac{1-\cos x}{x^2}\right| dx = 2\int_0^\infty \left|\frac{1-\cos x}{x^2}\right| dx$

$$= 2\int_0^1 \left|\frac{1-\cos x}{x^2}\right| dx + 2\int_1^\infty \left|\frac{1-\cos x}{x^2}\right| dx$$

$$\le 2\int_0^1 1\, dx + 2\int_1^\infty \frac{2}{x^2}\, dx < \infty \qquad \therefore \ \underset{\approx}{L^1}$$

$$\int_{-\infty}^{\infty} \frac{(1-\cos x)^2}{x^4}\, dx = 2\int_0^1 \frac{(1-\cos x)^2}{x^4}\, dx + 2\int_1^\infty \frac{(1-\cos x)^2}{x^4}\, dx$$

$$\int_0^1 \frac{(1-\cos x)^2}{x^4}\, dx \le \int_0^1 1\, dx < \infty$$

$$\int_1^\infty \frac{(1-\cos x)^2}{x^4} \le \int_1^\infty \frac{2}{x^4}\, dx < \infty \qquad \therefore \ L^2,,$$

7.1.2. $\int_{-\infty}^{\infty} |f(x)|\, dx = 2\int_0^\infty x^{-p}\, dx = 2\cdot\frac{1}{1-p} x^{1-p}\Big|_0^\infty \to \infty$ $\qquad \frac{1}{2}<p<1.$

$\int_{-\infty}^{\infty} |f(x)|^2\, dx = 2\int_0^\infty x^{-2p}\, dx = 2\cdot\frac{1}{1-2p} x^{1-2p}\Big|_0^\infty \to \infty \quad (\because \frac{1}{2}<p),,$

For $x\neq 0$, define $u(x) = \chi_{[-1,1]}\frac{1}{|x|^p}$ $\quad v(x) = \chi_A \frac{1}{|x|^p}$ $\quad A = \mathbb{R}\setminus[-1,1]$.

Then $\int_{-\infty}^{\infty} \chi_{[-1,1]}\frac{1}{|x|^p}\, dx = 2\int_0^1 x^{-p}\, dx = \frac{2}{1-p}\cdot x^{1-p}\Big|_0^1 < \infty \Rightarrow L^1$

$$\int_{-\infty}^{\infty} \left|\chi_A \frac{1}{|x|^p}\right|^2 dx = 2\int_1^\infty x^{-2p}\, dx < \infty \quad (\because 1<2p<2) \Rightarrow L^2.$$