7.2.7

$f \in L^2$ By Plancherel Theorem $\hat{f} \in L^2$. $\Rightarrow \int |\hat{f}|^2 d\xi < \infty$

$f' \in L^2$ By Plancherel Theorem $\hat{f'} \in L^2$. Since $\hat{f'} = i\xi \hat{f}(\xi) \Rightarrow \int \xi^2 |\hat{f}(\xi)|^2 d\xi < \infty$

therefore $\int (1+\xi^2) |\hat{f}(\xi)|^2 d\xi < \infty$.

We want to prove $\hat{f} \in L^1$, which means $\int |\hat{f}(\xi)| d\xi < \infty$

$$\int |\hat{f}(\xi)| d\xi = \int |\hat{f}(\xi)| \frac{\sqrt{1+\xi^2}}{\sqrt{1+\xi^2}} d\xi = \int \sqrt{1+\xi^2} |\hat{f}(\xi)| \frac{1}{\sqrt{1+\xi^2}} d\xi$$

Remind: Cauchy - Schwarz inequality

$$\int f \bar{g} \, dx \leq \sqrt{\int |f|^2 dx} \sqrt{\int |g|^2 dx}$$

Let $f = \sqrt{1+\xi^2} |\hat{f}(\xi)|$ $\quad g = \frac{1}{\sqrt{1+\xi^2}} = \bar{g}$

$$\int |\hat{f}(\xi)| d\xi \leq \sqrt{\int (1+\xi^2) |\hat{f}(\xi)|^2 d\xi} \sqrt{\int \frac{1}{1+\xi^2} d\xi}$$

Since $\int (1+\xi^2) |\hat{f}(\xi)|^2 d\xi < \infty$ $\quad \int \frac{1}{1+\xi^2} d\xi = \arctan\xi \Big|_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

$\int |\hat{f}(\xi)| d\xi < +\infty. \Rightarrow \hat{f} \in L^1$

7.3.9

7.3.9 lets us verify translation and phase transition does not change the concentration of the function.f satisfies the hypotheses of Heisenberg's inequality so that

$$(\triangle_a\ f)(\triangle_\alpha) \geq 1/4 \text{ for } a, \alpha \in R,$$

where $\triangle_a\ f = \frac{\int (x-a)^2 |f(x)|^2 dx}{\int |f(x)|^2 dx}$

part a)

$$\begin{aligned}
&\triangle_0\ F = \frac{\int x^2 |F(x)|^2 dx}{\int |F(x)|^2 dx} = \frac{\int x^2 |f(x+a)|^2 dx}{\int |f(x+a)|^2 dx} \\
&= \frac{\int (x'-a)^2 |f(x')|^2 dx}{\int |f(x')|^2 dx} \text{ by substitution } x' = x + a \\
&= \triangle_a\ f
\end{aligned}$$

part b)

$$\begin{aligned}
&\triangle_0\ \hat{F} = \frac{\int \xi^2 |\hat{F}(\xi)|^2 d\xi}{\int |\hat{F}(\xi)|^2 d\xi} = \frac{\int \xi^2 |\hat{f}(\xi+\alpha)|^2 d\xi}{\int |\hat{f}(\xi+\alpha)|^2 d\xi} \\
&= \frac{\int (\xi'-\alpha)^2 |\hat{f}(\xi')|^2 d\xi'}{\int |\hat{f}(\xi')|^2 d\xi'} \text{ by substitution } \xi' = \xi + \alpha \\
&= \triangle_\alpha\ \hat{f}
\end{aligned}$$