Problem 1 (10 pts) Let $f(x) \in C[0,\pi] \cap PS[0,\pi]$ but $f(0) \neq f(\pi)$. Consider the expansion of f into the Fourier cosine series, the Fourier sine series, and the Fourier series using complex exponentials. Which of the three series has the fastest decay in the size (i.e., magnitude) of the coefficients? Explain your reasoning.

Answer: The decay of the coefficients of the Fourier cosine series is the fastest because the Fourier cosine expansion of such f is the same as the Fourier series expansion of the evenly-extended function

$$f_e(x) \stackrel{\Delta}{=} \begin{cases} f(x) & \text{for } 0 \le x \le \pi; \\ f(-x) & \text{for } -\pi \le x \le 0. \end{cases}$$

This function f_e is in $C[-\pi,\pi] \cap PS[-\pi,\pi]$ and 2π-periodic without any discontinuity. Therefore, a_n, the nth Fourier coefficient of f_e (i.e., the nth Fourier cosine coefficients of f) decays at the rate of $O(n^{-2})$.

On the other hand, the coefficients of the Fourier sine series expansion of f is the same as the Fourier series expansion of the odd extension of f, i.e.,

$$f_o(x) \stackrel{\Delta}{=} \begin{cases} f(x) & \text{for } 0 \le x \le \pi; \\ -f(-x) & \text{for } -\pi \le x \le 0. \end{cases}$$

This function is discontinuous at $x = 0$ unless $f(0) = 0$. Moreover, viewing f_o as a 2π-periodic function, it is also discontinuous at $x = \pi$, i.e., $f(\pi-) \neq f(\pi+)$ because $f(\pi-) = f(\pi)$ and $f(\pi+) = f(-\pi) = -f(\pi)$ unless $f(\pi) = 0$. Therefore, in general, f_o is discontinuous. Thus, its nth Fourier coefficient (i.e., the nth Fourier sine coefficient) decays only with the speed $O(n^{-1})$.

Finally, as for the Fourier series expansion of f over $[0,\pi]$ as a function of period π, since $f(0) \neq f(\pi)$, the π-periodic extension of f is discontinuous at $x = k\pi$, $k \in \mathbb{Z}$. Thus, the decay of the nth Fourier coefficient is $O(n^{-1})$, the same as that of the Fourier sine series expansion.

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Problem 2 (20 pts) Solve Laplace's equation on the 2D annulus $\Omega = \{(r\cos\theta, r\sin\theta) \mid 1 < r < 2, -\pi \le \theta \le \pi\}$ with the following boundary condition:

$$\begin{aligned} u(1,\theta) &= 0 \quad \text{for } -\pi \le \theta \le \pi; \\ u(2,\theta) &= \begin{cases} -1 & \text{if } -\pi < \theta < 0; \\ 1 & \text{if } 0 < \theta < \pi. \end{cases} \end{aligned}$$

Hint: Laplace's equation in the polar coordinates (r,θ) is:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Answer: Let us use the separation of variable, i.e., let us assume that $u(r,\theta) = R(r)\Theta(\theta)$. Plug this into the above equation gives us

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \nu^2,$$

where ν is a constant independent of (r,θ). Let us first solve $\Theta'' + \nu^2\Theta = 0$. Its boundary condition is simply the periodic boundary condition thinking about the domain shape. That is $\Theta(-\pi) = \Theta(\pi)$. With this periodic boundary condition, we must have $\nu = n \in \mathbb{Z}$ and

$$\Theta(\theta) = \Theta_n(\theta) = \mathrm{e}^{\mathrm{i}n\theta}.$$

Now, let us consider the other equation in R:

$$r^2R'' + rR' - n^2R = 0.$$

Since this is an Euler equation, we seek the solution of the form $R(r) = r^\lambda$ for some λ. Inserting this to the above equation gives us

$$\lambda(\lambda-1) + \lambda - n^2 = 0 \Longleftrightarrow \lambda^2 - n^2 = 0 \Longleftrightarrow \lambda = \pm n.$$

Thus, we have the solution of the form:

$$R(r) = \begin{cases} A_n r^n + B_n r^{-n} & \text{if } n \ne 0; \\ A_0 + B_0 \ln r & \text{if } n = 0. \end{cases}$$

Thus, the solution to Laplace's equation can be written as their linear superpositions, i.e.,

$$u(r,\theta) = A_0 + B_0\ln r + \sum_n{}' (A_n r^n + B_n r^{-n})\mathrm{e}^{\mathrm{i}n\theta},$$

where $\sum_n'$ means the summation over all $n \in \mathbb{Z}$ except $n = 0$. Now, we have to determine A_n, B_n using the boundary conditions.

$$u(1,\theta) = A_0 + \sum_n{}' (A_n + B_n)\mathrm{e}^{\mathrm{i}n\theta} = 0 \quad \text{for any } \theta.$$

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Therefore, we must have $A_0 = 0$ and $B_n = -A_n$ for all n. Now the solution is of the form

$$u(r,\theta) = B_0 \ln r + \sum_n{}' A_n (r^n - r^{-n}) \mathrm{e}^{\mathrm{i}n\theta}.$$

Let us now use the other boundary condition.

$$u(2,\theta) = B_0 \ln 2 + \sum_n{}' A_n (2^n - 2^{-n}) \mathrm{e}^{\mathrm{i}n\theta} = \begin{cases} -1 & \text{if } -\pi < \theta < 0; \\ 1 & \text{if } 0 < \theta < \pi. \end{cases}$$

The Fourier series expansion of the righthand side is:

$$\frac{\mathrm{i}}{\pi} \sum_n{}' \frac{1-(-1)^n}{n} \mathrm{e}^{\mathrm{i}n\theta}.$$

By matching these, we have

$$B_0 = 0, \quad A_n (2^n - 2^{-n}) = -\frac{\mathrm{i}}{\pi} \frac{1-(-1)^n}{n}.$$

Therefore, we have:

$$\begin{aligned} u(r,\theta) &= -\frac{\mathrm{i}}{\pi} \left\{ \sum_{n>0} \frac{r^n - r^{-n}}{2^n - 2^{-n}} \frac{1-(-1)^n}{n} \mathrm{e}^{\mathrm{i}n\theta} + \sum_{n>0} \frac{r^n - r^{-n}}{2^n - 2^{-n}} \frac{1-(-1)^{-n}}{-n} \mathrm{e}^{-\mathrm{i}n\theta} \right\} \\ &= -\frac{\mathrm{i}}{\pi} \sum_{n>0} \frac{r^n - r^{-n}}{2^n - 2^{-n}} \frac{1-(-1)^n}{n} 2\mathrm{i} \sin n\theta \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{2^n - 2^{-n}} \frac{1-(-1)^n}{n} \sin n\theta \\ &= \boxed{\frac{4}{\pi} \sum_{m=1}^{\infty} \frac{r^{2m-1} - r^{-2m+1}}{(2m-1)\left(2^{2m-1} - 2^{-2m+1}\right)} \sin(2m-1)\theta}. \end{aligned}$$

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Problem 3 (20 pts) Solve the following 1D heat equation with the inhomogeneous boundary condition and the simple initial condition.

$$u_t = ku_{xx} \quad 0 \le x \le \ell, \quad t \ge 0$$

with

$$u_x(0,t) = 100; \quad u(\ell,t) = 0; \quad u(x,0) = 100.$$

Answer: Consider first the steady state solution $u_0(x)$ to get rid of the inhomogeneous boundary condition. u_0 satisfies

$$u_0'' = 0 \quad u_0'(0) = 100 \quad u_0(\ell) = 0.$$

By integrating this ODE twice and with those boundary conditions, we get $u_0(x) = 100(x-\ell)$.

Now, consider the residual $v(x,t) = u(x,t) - u_0(x)$. This function $v(x,t)$ satisfies

$$v_t = kv_{xx}, \quad \text{with} \quad v_x(0,t) = 0; \quad v(\ell,t) = 0; \quad v(x,0) = 100 - u_0(x) = 100(-x+\ell+1).$$

Now, we use the separation of variables as usual. Let $v(x,t) = X(x)T(t)$. Inserting this to the heat equation, we get

$$XT' = kX''T \Longrightarrow \frac{T'}{kT} = \frac{X''}{X} = -\nu^2 \in \mathbb{R}.$$

Thus we have

$$X'' + \nu^2 X = 0 \quad \text{with} \quad X'(0) = 0; \quad X(\ell) = 0.$$

The solution of this ODE is of the form:

$$X(x) = A\cos\nu x + B\sin\nu x,$$

where A, B are some constants. (The other cases of But because of the boundary conditions, we have

$$X'(0) = \nu B = 0 \quad X(\ell) = A\cos\nu\ell = 0.$$

Thus we must have

$$\nu\ell = \frac{\pi}{2} + n\pi = \left(n + \frac{1}{2}\right)\pi \quad n \in \mathbb{Z}.$$

Thus the eigenvalues are

$$\nu_n = \frac{\left(n+\frac{1}{2}\right)\pi}{\ell}.$$

At this point, n can be any integer. However, we can show only nonnegative integers are allowed as n below.

We can now solve the equation $T' = -k\nu_n^2 T$ easily as

$$T(t) = \mathrm{e}^{-k\nu_n^2 t} = \exp\left\{-k\left(\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}\right)^2 t\right\}.$$

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In order to prevent the blow up of the heat distribution, we must have $n \geq 0$. Thus, the eigenvalues are Thus the desired eigenvalues are

$$\boxed{\nu_n = \frac{\left(n+\frac{1}{2}\right)\pi}{\ell} \qquad n = 0, 1, \ldots}$$

Now, using the superposition principle, we can write the solution $v(x,t)$ as

$$v(x,t) = \sum_{n=0}^{\infty} a_n \exp\left\{-k\left(\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}\right)^2 t\right\} \cos\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}x,$$

where the coefficients $\{a_n\}$ must be determined to match with the initial condition, i.e.,

$$v(x,0) = \sum_{n=0}^{\infty} a_n \cos\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}x = 100(-x+\ell+1).$$

In other words, a_n is simply the Fourier cosine coefficients over $[0,\ell]$ of the function $100(-x+\ell+1)$. Hence, we have

$$\begin{aligned}
a_n &= \frac{2}{\ell}\int_0^{\ell} 100(-x+\ell+1)\cos\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}x\,\mathrm{d}x \\
&= \frac{200}{\ell}\int_0^{\ell} (-x+\ell+1)\cos\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}x\,\mathrm{d}x \\
&= \frac{200}{\ell}\left\{\left[\frac{\sin\left(n+\frac{1}{2}\right)\pi x/\ell}{\left(n+\frac{1}{2}\right)\pi/\ell}(-x+\ell+1)\right]_0^{\ell} + \frac{1}{\left(n+\frac{1}{2}\right)\pi/\ell}\int_0^{\ell}\sin\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}x\,\mathrm{d}x\right\} \\
&= \frac{200}{\ell}\left\{\frac{(-1)^n}{\left(n+\frac{1}{2}\right)\pi/\ell} + \frac{1}{\left(\left(n+\frac{1}{2}\right)\pi/\ell\right)^2}\right\} \\
&= \frac{200}{\left(n+\frac{1}{2}\right)\pi}\left\{(-1)^n + \frac{\ell}{\left(n+\frac{1}{2}\right)\pi}\right\}.
\end{aligned}$$

Thus, we have

$$v(x,t) = 200\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)\pi}\left\{(-1)^n + \frac{\ell}{\left(n+\frac{1}{2}\right)\pi}\right\}\exp\left\{-k\left(\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}\right)^2 t\right\}\cos\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}x.$$

Finally, we have the following solution

$$u(x,t) = u_0(x) + v(x,t) = \boxed{100(x-\ell) + 200\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)\pi}\left\{(-1)^n + \frac{\ell}{\left(n+\frac{1}{2}\right)\pi}\right\}\exp\left\{-k\left(\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}\right)^2 t\right\}\cos\frac{\left(n+\frac{1}{2}\right)\pi}{\ell}x.}$$

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Problem 4 (10 pts) Let $[0,1] \subset \mathbb{R}$ be a unit interval.

(a) (5 pts) Prove that
$$L^2[0,1] \subset L^1[0,1].$$

Hint: Use the Cauchy-Schwarz inequality!

Answer: Take any $f \in L^1[0,1]$. Then consider
$$\|f\|_1 = \int_0^1 |f(x)|\,\mathrm{d}x.$$

Our objective is to prove $\|f\|_1 < \infty$ if $f \in L^2[0,1]$. To do so, we can use the Cauchy-Schwarz inequality as follows.
$$\int_0^1 |f(x)|\,\mathrm{d}x = \int_0^1 1 \cdot |f(x)|\,\mathrm{d}x \le \sqrt{\int_0^1 1^2\,\mathrm{d}x} \cdot \sqrt{\int_0^1 |f(x)|^2\,\mathrm{d}x} = \sqrt{\int_0^1 |f(x)|^2\,\mathrm{d}x} = \|f\|_2 < \infty$$

since $f \in L^2[0,1]$. □

(b) (5 pts) Give an example of a function $f(x)$ that belongs to $L^1[0,1]$ but not to $L^2[0,1]$.

Answer: We need to come up with a function $f(x)$ such that $\int_0^1 |f(x)|\,\mathrm{d}x < \infty$ but $\int_0^1 |f(x)|^2\,\mathrm{d}x = infty$. So, the easiest example is $\boxed{f(x) = \frac{1}{\sqrt{x}}}$. In fact,
$$\int_0^1 |f(x)|\,\mathrm{d}x = \int_0^1 x^{1/2}\,\mathrm{d}x = \left[2x^{1/2}\right]_0^1 = 2$$

while
$$\int_0^1 |f(x)|^2\,\mathrm{d}x = \int_0^1 \frac{1}{x}\,\mathrm{d}x = [\ln x]_0^1 = \infty.$$

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Problem 5 (20 pts) Prove the following basic Fourier transform properties:

(a) (5 pts) For any $f \in L^1$,

$$\widehat{\tau_a f}(\xi) = \mathrm{e}^{-2\pi \mathrm{i}\xi a}\hat{f}(\xi),$$

where τ_a is the translation (i.e., shift) operator on $\mathbb{R}$, and $a \in \mathbb{R}$.

Answer:

$$\begin{aligned}
\widehat{\tau_a f}(\xi) &= \int f(x-a)\mathrm{e}^{-2\pi \mathrm{i}\xi x}\,\mathrm{d}x \\
&= \int f(y)\mathrm{e}^{-2\pi \mathrm{i}\xi(y+a)}\,\mathrm{d}y \quad \text{via } y = x-a; \\
&= \mathrm{e}^{-2\pi \mathrm{i}\xi a}\int f(y)\mathrm{e}^{-2\pi \mathrm{i}\xi y}\,\mathrm{d}y \\
&= \mathrm{e}^{-2\pi \mathrm{i}\xi a}\hat{f}(\xi).
\end{aligned}$$

(b) (5 pts) For any $f \in L^2$,

$$\widehat{\delta_s f}(\xi) = \delta_{1/s}\hat{f}(\xi),$$

where δ_s is the dilation (i.e., scaling) operator on $\mathbb{R}$, i.e.,

$$\delta_s f(x) \triangleq \frac{1}{\sqrt{s}} f\left(\frac{x}{s}\right) \quad \text{for } s > 0.$$

Answer:

$$\begin{aligned}
\widehat{\delta_s f}(\xi) &= \int \frac{1}{\sqrt{s}} f\left(\frac{x}{s}\right)\mathrm{e}^{-2\pi \mathrm{i}\xi x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{s}}\int f(y)\mathrm{e}^{-2\pi \mathrm{i}\xi s y}\,\mathrm{d}sy \quad \text{via } y = x/s; \\
&= \sqrt{s}\int f(y)\mathrm{e}^{-2\pi \mathrm{i}s\xi y}\,\mathrm{d}y \\
&= \sqrt{s}\hat{f}(s\xi) \\
&= \delta_{1/s}\hat{f}(\xi).
\end{aligned}$$

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(c) (5 pts) Let

$$g(x;\mu,\sigma) \triangleq \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Suppose we know that

$$\mathcal{F}\left\{g\left(x;0,\frac{1}{\sqrt{2\pi}}\right)\right\} = \mathcal{F}\left\{e^{-\pi x^2}\right\} = e^{-\pi\xi^2}.$$

Then, using the formula (a) and (b) prove

$$\mathcal{F}\{g(x;\mu,\sigma)\} = e^{-2\pi(i\mu\xi+\pi\sigma^2\xi^2)}.$$

Answer: First of all, it is easy to see

$$g(x;\mu,\sigma) = \tau_\mu g(x;0,\sigma).$$

Now, using the dilation operator, notice that

$$\delta_{2\pi\sigma^2} e^{-\pi x^2} = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}.$$

Hence, we have

$$g(x;\mu,\sigma) = \tau_\mu \delta_{2\pi\sigma^2} e^{-\pi x^2}.$$

Taking the Fourier transform and using the formulas in (a), (b), we have

$$\mathcal{F}\{g(x;\mu,\sigma)\} = e^{-2\pi i\xi\mu}\, \delta_{\frac{1}{2\pi\sigma^2}}\, e^{-\pi\xi^2} = e^{-2\pi(i\mu\xi+\pi\sigma^2\xi^2)}.$$

□

(d) (5 pts) Assuming that $f \in C(\mathbb{R}) \cap PS(\mathbb{R})$ and $f, f' \in L^1$,

$$\widehat{f'}(\xi) = 2\pi i\xi \hat{f}(\xi).$$

Answer:

$$\begin{aligned}
\widehat{f'}(\xi) &= \int_{-\infty}^{\infty} f'(x) e^{-2\pi i\xi x}\, dx \\
&= \left[f(x)e^{-2\pi i\xi x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-2\pi i\xi) e^{-2\pi i\xi x}\, dx \\
&\overset{(*)}{=} 2\pi i\xi \int_{-\infty}^{\infty} f(x) e^{-2\pi i\xi x}\, dx \\
&= 2\pi i\xi \hat{f}(\xi).
\end{aligned}$$

The equality (*) above is valid because $f \to 0$ as $x \to \pm\infty$.

Problem 6 (20 pts)

(a) (7 pts) Compute the Fourier transform of $f(x) = \chi_{(-\frac{1}{2},\frac{1}{2})}(x)$, i.e., the characteristic function (also known as the indicator function) of the 1D interval $\left(-\frac{1}{2},\frac{1}{2}\right)$.

Answer: This is an easy exercise.

$$\begin{aligned}
\hat{f}(\xi) &= \int \chi_{(-\frac{1}{2},\frac{1}{2})}(x)\mathrm{e}^{-2\pi\mathrm{i}\xi x}\,\mathrm{d}x \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{-2\pi\mathrm{i}\xi x}\,\mathrm{d}x \\
&= \left[\frac{\mathrm{e}^{-2\pi\mathrm{i}\xi x}}{-2\pi\mathrm{i}\xi}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{\mathrm{e}^{-\pi\mathrm{i}\xi} - \mathrm{e}^{\pi\mathrm{i}\xi}}{-2\pi\mathrm{i}\xi} \\
&= \boxed{\frac{\sin\pi\xi}{\pi\xi}.}
\end{aligned}$$

(b) (7 pts) State *Heisenberg's inequality* for $f \in L^2$. Note that you need to define all the necessary quantities to state this inequality precisely. Also state when the *equality* holds.

Answer: Let us define the *spread* of such f about $x = a$ as

$$\Delta_a f \triangleq \frac{\int (x-a)^2 |f(x)|^2\,\mathrm{d}x}{\|f\|_2^2}.$$

Then, Heisenberg's inequality is stated as

$$\left(\Delta_{x_0} f\right)\left(\Delta_{\xi_0}\hat{f}\right) \geq \frac{1}{16\pi^2},$$

where x_0, ξ_0 are arbitrary constants in $\mathbb{R}$, and the equality holds if and only if f is a constant multiple of a Gaussian function.

(c) (6 pts) Check whether f and $\hat{f}$ in Part (a) satisfies Heisenberg's inequality.

Answer: Let's compute $\Delta_{x_0} f$ and $\Delta_{\xi_0} \hat{f}$. But first of all, let's compute $\|f\|_2$.

$$\|f\|_2^2 = \int_{-\infty}^{\infty} \chi_{(-\frac{1}{2},\frac{1}{2})}(x)\,\mathrm{d}x \int_{-\frac{1}{2}}^{\frac{1}{2}} 1\,\mathrm{d}x = 1.$$

So, $\|f\|_2 = 1$. Using the Plancherel equality, we also have $\|\hat{f}\|_2 = \|f\|_2 = 1$. Since $\|\hat{f}\|_2 = \|f\|_2 = 1$, we do not have to worry about the normalization by them.

$$\begin{aligned}
\Delta_{x_0} f &= \int_{-\infty}^{\infty} (x - x_0)^2 |f(x)|^2 \,\mathrm{d}x / \|f\|_2^2 \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} (x - x_0)^2 \,\mathrm{d}x \\
&= \left[\frac{(x - x_0)^3}{3} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= x_0^2 + \frac{1}{12}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Delta_{\xi_0} \hat{f} &= \int_{-\infty}^{\infty} (\xi - \xi_0)^2 |\hat{f}(x)|^2 \,\mathrm{d}\xi / \|\hat{f}\|_2^2 \\
&= \int_{-\infty}^{\infty} (\xi - \xi_0)^2 \left(\frac{\sin \pi\xi}{\pi\xi} \right)^2 \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \xi^2 \left(\frac{\sin \pi\xi}{\pi\xi} \right)^2 \mathrm{d}x - 2\xi_0 \int_{-\infty}^{\infty} \frac{\sin^2 \pi\xi}{\pi\xi} \,\mathrm{d}x + \xi_0^2 \int_{-\infty}^{\infty} \left(\frac{\sin \pi\xi}{\pi\xi} \right)^2 \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{\sin^2 \pi\xi}{\pi^2} \,\mathrm{d}x - 2\xi_0 \cdot 0 + \xi_0^2 \cdot 1 \\
&= \infty.
\end{aligned}$$

So, even though $\Delta_{x_0} f$ is finite, $\Delta_{\xi_0} \hat{f}$ diverges to ∞. So, Heisenberg's inequality still holds. $\square$

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