(3.3.2) Suppose $f_n \to f$ in norm.

<u>To show</u>: $\big| \|f_n\| - \|f\| \big| \to 0$ as $n \to \infty$.

First show that $\big| \|f\| - \|g\| \big| \le \|f-g\|$, for any $g \in L^2(a,b)$

Consider case $\|f\| \ge \|g\|$. Then

$\|f\| = \|f+g-g\| \le \|f-g\| + \|g\|$, by Triangle Ineq.

$\Rightarrow \quad \|f\| - \|g\| \le \|f-g\|$.

Similarly for case $\|g\| \ge \|f\|$,

$$\|g\| = \|g-f+f\| \le \underbrace{\|g-f\|}_{=\|f-g\|} + \|f\|$$

$\Rightarrow \|g\| - \|f\| \le \|f-g\|$.

In both cases, $\big| \|f\| - \|g\| \big| \le \|f-g\|$.

Now, substitute $\|f_n\|$ in place of $\|g\|$:

$\big| \|f_n\| - \|f\| \big| \le \|f_n - f\| \to 0$ as $n \to \infty$, since $f_n \to f$ in norm.

3.3.3.

Proof. Let $f \in PC[a,b]$ be discontinuous at $x_0 \in (a,b)$ without loss of generality. Define $f_n : [a,b] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} f(x), & x_0 + r_n \le x \le b\,; \\ g_n(x), & x_0 - r_n \le x \le x_0 + r_n; \\ f(x), & a \le x \le x_0 - r_n \end{cases}$$

where $r_n \to 0$ as $n \to \infty$ and $g_n(x)$ is any continuous function. Then

$$\begin{aligned} \| f_n - f \|^2 &= \int_{x_0 - r_n}^{x_0 + r_n} | g_n(x) - f(x) |^2 \, dx \\ &= \int_{x_0 - r_n}^{x_0} | g_n(x) - f(x) |^2 \, dx + \int_{x_0}^{x_0 + r_n} | g_n(x) - f(x) |^2 \, dx \\ &\le r_n A \end{aligned}$$

since $| g_n(x) - f(x) |^2$ is bounded on each interval. Thus, $f_n \to f$ in the norm as $n \to \infty$. $\square$