(3.3.4) Let $y = cx + d$, so $dy = c\,dx$, then

$$\langle \psi_m, \psi_n \rangle = \int_{(a-d)/c}^{(b-d)/c} \phi_m(cx+d)\,\phi_n(cx+d)\,dx$$

$$= \int_a^b \phi_m(y)\,\phi_n(y)\,dy = \begin{cases} 1, & \text{if } m=n \\ 0, & \text{if } m \neq n \end{cases}$$

$\therefore$ $\{\psi_m\}$ is orthonormal set in $L^2(\frac{a-d}{c}, \frac{b-d}{c})$

To complete the argument, we show property (a) in Thm 3.4 is satisfied.
Suppose $f \in L^2(\frac{a-d}{c}, \frac{b-d}{c})$ is such that $\langle f, \psi_n \rangle = 0$ for all n. We need to show $f = 0$.

$$\langle f, \psi_n \rangle = \sqrt{c}\int_{(a-d)/c}^{(b-d)/c} f(x)\,\phi_n(cx+d)\,dx$$

$$= \int_a^b \frac{1}{\sqrt{c}} f\left(\frac{y-d}{c}\right) \phi_n(y)\,dy = 0, \quad \forall n.$$

Since $\{\phi_n\}$ is orthonormal basis, the function $\frac{1}{\sqrt{c}} f(\frac{y-d}{c}) = 0$ for a.e. y in $[a,b]$.

Hence $f = 0$ for a.e. y in $[\frac{a-d}{c}, \frac{b-d}{c}]$.

$\therefore$ $\{\psi_n\}$ is orthonormal basis in $L^2(\frac{a-d}{c}, \frac{b-d}{c})$

(3.3.9) Suppose $\{\phi_n\}_1^\infty$ is orthonormal basis for $L^2(a,b)$

Let $f, g \in L^2(a,b)$.

Then $f = \sum_1^\infty \langle f, \phi_n \rangle \phi_n$, convergence in norm.

<u>To show</u>: $\sum_{n=1}^N \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle} \longrightarrow \langle f, g \rangle$ as $N \to \infty$.

$$\sum_1^N \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle} = \sum_1^N \langle f, \phi_n \rangle \langle \phi_n, g \rangle = \left\langle \sum_1^N \langle f, \phi_n \rangle \phi_n, g \right\rangle$$

$$\xrightarrow[N\to\infty]{} \left\langle \sum_1^\infty \langle f, \phi_n \rangle \phi_n, g \right\rangle = \langle f, g \rangle,$$

by problem 1.

Extra Problem: Prove Cauchy-Schwarz inequality

Proof. $0 \le \int_a^b |f(x) - \delta g(x)|^2 dx$
$\Rightarrow 0 \le \int_a^b |f(x)|^2 dx + |\delta|^2 \int_a^b |g(x)|^2 dx - \overline{\delta} \int_a^b f(x)\overline{g(x)} dx - \delta \int_a^b \overline{f(x)} g(x) dx$.
Without loss of generality, can assume $\int_a^b |g(x)|^2 dx > 0$. Take $\delta = \frac{\int_a^b f(x)\overline{g(x)}dx}{\int_a^b |g(x)|^2 dx}$, conclude that $0 \le \int_a^b |f(x)|^2 dx + |\int_a^b f(x)\overline{g(x)} dx|^2 / \int_a^b |g(x)|^2 dx$. Reorganize the terms, we get the result. $\square$