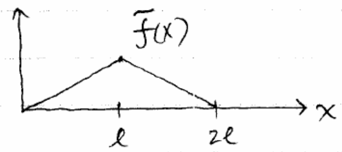
(3.3.6)

(a) $\left\{ \sqrt{\frac{2}{\pi}} \sin kx \right\}_1^\infty$ is orthonormal basis for $L^2(0,\pi)$.

Apply Prob 4 with $a=0$, $d=0$, $b=\pi$, $c=\frac{\pi}{2\ell}$, then we get $\left\{ \varphi_k(x) = \frac{1}{\sqrt{\ell}} \sin\left(\frac{\pi k x}{\ell}\right) \right\}$ an orthonormal basis for $L^2(0,2\ell)$.

(b) Extend f, i.e. $\tilde{f}(x) = \tilde{f}(2\ell - x) = f(x)$ for $x \in [0,\ell]$

EX:



$$\langle f, \varphi_{2n} \rangle = \int_0^{2\ell} \tilde{f}(x) \frac{1}{\sqrt{\ell}} \sin\left(\frac{n\pi x}{\ell}\right) dx = \int_0^{\ell} \tilde{f}(x) \frac{1}{\sqrt{\ell}} \sin\left(\frac{n\pi x}{\ell}\right) dx + \int_\ell^{2\ell} \tilde{f}(x) \frac{1}{\sqrt{\ell}} \sin\left(\frac{n\pi x}{\ell}\right) dx$$

let $y = 2\ell - x$, then

$$\int_\ell^{2\ell} \tilde{f}(x) \frac{1}{\sqrt{\ell}} \sin\left(\frac{n\pi x}{\ell}\right) dx = -\int_\ell^{0} \tilde{f}(2\ell - y) \frac{1}{\sqrt{\ell}} \sin\left(\frac{n\pi}{\ell}(2\ell - y)\right) dy$$

$$= \int_0^{\ell} \tilde{f}(y) \frac{1}{\sqrt{\ell}} \cdot \left(-\sin\left(\frac{n\pi y}{\ell}\right)\right) dy,$$

since $\tilde{f}(x) = \tilde{f}(2\ell - x)$ and $\sin(2n\pi - \theta) = -\sin\theta$.

$\therefore \langle f, \varphi_{2n} \rangle = 0$.

(3.3.6) (cont.)

(b) Repeat the steps

$$\langle \tilde{f}, \psi_{2n-1} \rangle = \frac{1}{\sqrt{\ell}} \int_0^{2\ell} \tilde{f}(x) \sin\left((2n-1)\frac{\pi x}{2\ell}\right) dx$$

$$= \frac{1}{\sqrt{\ell}} \int_0^{\ell} \tilde{f}(x) \sin\left((2n-1)\frac{\pi x}{2\ell}\right) dx + \frac{1}{\sqrt{\ell}} \int_0^{\ell} \tilde{f}(2\ell-y) \sin\left((2n-1)\pi - (2n-1)\frac{\pi y}{2\ell}\right) d$$

$$= \frac{2}{\sqrt{\ell}} \int_0^{\ell} f(x) \sin\left((2n-1)\frac{\pi x}{2\ell}\right) dx \ , \quad \text{since } \sin((2n-1)\pi - \theta) = \sin\theta$$

$$\text{and } \tilde{f}(x) = \hat{f}(2\ell - x) = f(x$$

$$= \sqrt{2} \langle f, \phi_n \rangle .$$

(c) Suppose $\langle f, \phi_n \rangle = 0 \ \forall n$. This means

$\langle \tilde{f}, \psi_{2n-1} \rangle = \sqrt{2} \langle f, \phi_n \rangle = 0$, so $\langle \tilde{f}, \psi_n \rangle = 0 \ \forall n$.

Since $\{\psi_n\}$ is an O.N. basis for $L^2(0, 2\ell)$,

$\tilde{f} = 0 \Rightarrow f = 0$.

To conclude, $\{\phi_n\}$ is an O.N. basis for $L^2(0,\ell)$ by property (a) of Thm 3.4.

(3.3.8) $f(x) = 1 = \sum_1^{\infty} \langle f, \phi_n \rangle \phi_n$, $g(x) = x = \sum_1^{\infty} \langle g, \phi_n \rangle \phi_n$.

$$\langle f, \phi_n \rangle = \int_0^{\ell} \sqrt{\tfrac{2}{\ell}} \sin\left(n - \tfrac{1}{2}\right)\frac{\pi x}{\ell}\, dx$$

$$= -\sqrt{\tfrac{2}{\ell}} \cdot \frac{2\ell}{\pi(2n-1)} \cos\left(n - \tfrac{1}{2}\right)\frac{\pi x}{\ell} \Big|_0^{\ell}$$

$$= \sqrt{\tfrac{2}{\ell}} \cdot \frac{2\ell}{\pi(2n-1)} = \frac{2\sqrt{2\ell}}{\pi(2n-1)} .$$

To find $\langle g, \phi_n \rangle$, just integrate by parts .