3.4.2. $\langle f_0 . f_1 \rangle = \int_0^\infty (ax+b) . e^{-x} dx = a\int_0^\infty x e^{-x} dx + b\int_0^\infty e^{-x} dx = a+b = 0$,,

$\langle f_0 . f_2 \rangle = \int_0^\infty (Ax^2 + Bx + C) e^{-x} dx = A \cdot 2! + B + C = 0.$ $\cdots$ ①

~~$\langle f_1 . f_2 \rangle =$~~

$\| f_1 \|^2 = \int_0^\infty (ax-a)^2 \cdot dx = \int_0^\infty (a^2x^2 - 2a^2 x \cdot + a^2) . e^{-x} dx$

$= a^2 \cdot 2! - 2a^2 \cdot 1! + a^2 = a^2 = 1.$ $\therefore a = \pm 1.$

Let $a=1$, then $b=-1$. $\therefore f_1(x) = x-1.$

$\langle f_1 . f_2 \rangle = \int_0^\infty (Ax^2 + Bx + C) \cdot (x-1) \cdot e^{-x} dx$

$= \int_0^\infty (Ax^3 + Bx^2 + Cx - Ax^2 - Bx - C) e^{-x} dx$

$= A \cdot 3! + (B-A) 2! + (C-B) 1! - C$

$= 4A + B = 0.$ $B = -4A.$ $\cdots$ ②

$\| f_2 \|^2 = \int_0^\infty (Ax^2 - 4Ax + 2A)^2 e^{-x} dx$ using ① and ②

$= 4A^2. = 1$ $A = \pm \frac{1}{2}.$

Choosing $A = \frac{1}{2}$, $B = -2$ and $C = 1$.

$\therefore f_1(x) = x-1$

$f_2(x) = \frac{1}{2}x^2 - 2x + 1.$

3.4.3. $f_n = (x+iy)^n = r^n e^{in\theta}$

$f_m = (x+iy)^m = r^m e^{im\theta}$

$\langle f_n . f_m \rangle = \int_0^{2\pi} \int_0^1 r^n e^{in\theta} . r^m e^{-im\theta} \, r dr d\theta$

$= \int_0^{2\pi} e^{i\theta(n-m)} \int_0^1 r^{n+m+1} dr \, d\theta.$

$= \frac{1}{n+m+2} \int_0^{2\pi} e^{i\theta(n-m)} d\theta = \frac{1}{i(n-m)(n+m+2)} \left(e^{2\pi i(n-m)} - 1 \right)$ if $n \neq m$

$= 0.$

$\| f_n \|^2 = \int_0^{2\pi} \int_0^1 r^{2n+1} e^{i\theta n} e^{-in\theta} dr d\theta.$

$= \frac{1}{2n+2} r^{2n+2} \Big|_0^1 . \int_0^{2\pi} d\theta = \frac{2\pi}{2n+2} . = \frac{\pi}{n+1} .$ $\therefore \| f_n \| = \left(\frac{\pi}{n+1} \right)^{\frac{1}{2}}.$