Problem 3.4.6 If f_n is a sequence in $L^2(a,b)$ where $a > -\infty, b < \infty$, $f_n \to 0$ uniformly implies that $f_n \to 0$ in norm. However, if $a = -\infty$ or $b = \infty$, it's not true. Let

$$f_n(x) = \frac{1}{n} \text{ when } 0 < x \leq n^2$$

Otherwise, $f_n = 0$. Since $\sup|f_n(x)| = \frac{1}{n} \to 0$ when $n \to \infty$, $f_n \to 0$ uniformly. However, $||f||_{L^2(0,\infty)} = \int_0^{n^2} \frac{1}{n^2} dx = 1 \nrightarrow 0$ when $n \to \infty$, which means $f_n \nrightarrow 0$ in norm.

3.4.7. (a) Using Corollary 3.1., we need to find an orthonormal ~~set~~ set for $L^2[0,\pi]$.
Normalizing $\{1, \cos x, \cos 2x\}$, we get $\{\frac{1}{\sqrt{\pi}}, \frac{\sqrt{2}}{\sqrt{\pi}}\cos x, \frac{\sqrt{2}}{\sqrt{\pi}}\cos 2x\}$.
Thus, the best approx. in norm to the x on $[0,\pi]$ is

$$\langle x, \tfrac{1}{\sqrt{\pi}}\rangle \tfrac{1}{\sqrt{\pi}} + \langle x, \tfrac{\sqrt{2}}{\sqrt{\pi}}\cos x\rangle \tfrac{\sqrt{2}}{\sqrt{\pi}}\cos x + \langle x, \tfrac{\sqrt{2}}{\sqrt{\pi}}\cos 2x\rangle \tfrac{\sqrt{2}}{\sqrt{\pi}}\cos 2x$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\cos x.$$

(b) In the same way as in (a)

$$\langle x, \tfrac{\sqrt{2}}{\sqrt{\pi}}\sin x\rangle \tfrac{\sqrt{2}}{\sqrt{\pi}}\sin x + \langle x, \tfrac{\sqrt{2}}{\sqrt{\pi}}\sin 2x\rangle \tfrac{\sqrt{2}}{\sqrt{\pi}}\sin 2x$$

$$= 2\sin x - \sin 2x$$

(c). In the same way as in (a)

$$f \approx -\frac{4}{\pi}\cos x + 2\sin x.$$