Homework 14.

3.5.3 $f'' + \lambda f = 0$, $f(0) = 0$, $f'(l) = 0$ on $[0, l]$.

Let $f(x) = e^{mx}$ as a trial solution.

$$m^2 e^{mx} + \lambda e^{mx} = 0.$$

$$m = \pm\sqrt{-\lambda}.$$

i) $\lambda = 0$.

Then, $f(x) = ax + b$

By applying B.C., $a \cdot 0 + b = 0$.

$a = 0$.

$\therefore$ We get the trivial solution.

ii) $\lambda < 0$.

$f(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$f(0) = c_1 + c_2 = 0.$ $c_1 = -c_2$

$f'(x) = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x} + c_2(-\sqrt{-\lambda})e^{-\sqrt{-\lambda}x}$

$f'(l) = c_1\sqrt{-\lambda}(e^{\sqrt{-\lambda}\cdot l} + e^{-\sqrt{-\lambda}\cdot l}) = 0.$

$\therefore$ $c_1 = 0. = c_2.$

$\therefore$ We get the trivial solution.

iii) $\lambda > 0$.

$f(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$

$f(0) = c_1 + 0 = 0.$

$f'(x) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}x.$ $f'(l) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}\cdot l = 0.$

$\cos\sqrt{\lambda}\cdot l = 0$ $(\because \lambda \neq 0)$

$\Rightarrow$ $\sqrt{\lambda}\cdot l = (n+\frac{1}{2})\pi.$

$\therefore$ $\lambda = (n+\frac{1}{2})^2\pi^2 \cdot \frac{1}{l^2}.$

$\therefore$ Eigenvalues : $\lambda_n = (n+\frac{1}{2})^2\frac{\pi^2}{l^2}.$ $n \in \mathbb{Z}.$

Eigenfunctions : $f_n(x) = \sin(n+\frac{1}{2})\frac{\pi}{l}\cdot x.$

Normalizing, $\int_0^l |f_n(x)|^2 dx = \frac{l}{2}$, using $\sin^2 x = \frac{1-\cos 2x}{2}$.

$\therefore$ Normalized eigenfunctions $\sqrt{\frac{2}{l}}\cdot\sin(n+\frac{1}{2})\frac{\pi}{l}x.$

3.5.7. $f'' + \lambda f = 0$. $f(0) = 0$ $f'(1) = -f(1)$.

In the same way as in 3.5.3.

For $\lambda = 0$, we get the trivial solution.

For $\lambda < 0$, $f(0) = c_1 + c_2 = 0$.

$$f'(1) = c_1\sqrt{-\lambda}\,(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}})$$
$$= -c_1 e^{\sqrt{\lambda}} + c_1 e^{-\sqrt{\lambda}}.$$
$$\Rightarrow -c_1(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}) = c_1\sqrt{-\lambda}\,(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}})$$

If $c_1 \neq 0$, $\dfrac{1-\sqrt{-\lambda}}{1+\sqrt{-\lambda}} = e^{2\sqrt{\lambda}}$ which does not have a solution.

Thus, $c_1 = 0$

For $\lambda > 0$, $f(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$.

$f(0) = c_1 = 0$.

$f'(x) = \sqrt{\lambda}\,c_2\cos\sqrt{\lambda}x$, so $c_2\sin\sqrt{\lambda} + c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0$.

Eigenvalues are $\lambda_n = \nu_n^2$ where the ν_n's are positive solutions of $\tan\nu = -\nu$ "

Eigenfunctions are $\phi_n(x) = c_n\sin\nu_n x$. and

$c_n = \left(\dfrac{2}{1+\cos^2\alpha_n}\right)^{\frac{1}{2}}$ to normalize.