3.5.10. $(xf')' + \lambda x^{-1} f = 0 \qquad f(1) = f(b) = 0. \qquad b > 1.$

sol. $xf'' + f' + \lambda x^{-1} f = 0 \quad \Rightarrow \quad x^2 f''(x) + x f'(x) + \lambda \cdot f(x) = 0$; Euler equation.

Then $f(x) = c_1 x^{r_1} + c_2 x^{r_2}$ where r_1 and r_2 are roots of

$r(r-1) + r + \lambda = 0. \qquad$ if $r_1 \neq r_2$

$r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$

i) $\lambda = 0$; $f(x) = C$. Imposing $f(1) = f(b) = 0$, $f(x) = 0$; trivial solution.

ii) $\lambda < 0$; $r_1, r_2 \in \mathbb{R}$.

$\Rightarrow f(x) = c_1 x^{\sqrt{-\lambda}} + c_2 x^{-\sqrt{-\lambda}}$

$f(1) = c_1 + c_2 = 0 \qquad c_1 = -c_2$

$f(b) = c_1 b^{\sqrt{-\lambda}} - c_1 b^{-\sqrt{-\lambda}} = 0$

$\therefore c_1 = 0$ or $b^{\sqrt{-\lambda}} = b^{-\sqrt{-\lambda}} \Rightarrow \lambda = 0$; trivial solution

iii) $\lambda > 0$; $f(x) = c_1 x^{i\sqrt{\lambda}} + c_2 x^{-i\sqrt{\lambda}} = c_1 e^{i\sqrt{\lambda}\log x} + c_2 e^{-i\sqrt{\lambda}\log x}$

$f(1) = c_1 + c_2 = 0 \quad ; c_1 = -c_2.$

$f(b) = c_1 e^{i\sqrt{\lambda}\log b} - c_1 e^{-i\sqrt{\lambda}\log b} = 0$

$\Rightarrow e^{i\sqrt{\lambda}\log b} = e^{-i\sqrt{\lambda}\log b}$ to avoid the trivial solution.

$\Rightarrow i\sqrt{\lambda}\log b = -i\sqrt{\lambda}\log b + 2n\pi i$

$\sqrt{\lambda}\log b = -\sqrt{\lambda}\log b + 2n\pi$

$\therefore \sqrt{\lambda} = \dfrac{n\pi}{\log b} \quad \Rightarrow \quad \lambda_n = \left(\dfrac{n\pi}{\log b}\right)^2$; eigenvalues.

Now, $f(x) = c_1(e^{i\sqrt{\lambda}\log x} - e^{-i\sqrt{\lambda}\log x})$

$= 2ic_1 \sin(\sqrt{\lambda}\log x)$

$= 2ic_1 \sin\left(\dfrac{n\pi}{\log b}\cdot\log x\right) = C\sin\left(\dfrac{n\pi}{\log b}\log x\right)$; eigenfunctions

Normalizing, $\|f_n\|^2 = \int_1^b \sin^2\left(\dfrac{n\pi}{\log b}\cdot\log x\right)\cdot\dfrac{1}{x}\,dx$

$= \dfrac{\log b}{2}$ by letting $\frac{1}{x} = u$ and $\sin^2 x = \frac{1-\cos 2x}{2}$

$\therefore$ Normalized eigenfunctions:

$f_n(x) = \sqrt{\dfrac{2}{\log b}}\cdot\sin\left(\dfrac{n\pi}{\log b}\cdot\log x\right)$

3.5.12.

f satisfies. $(rf')' + pf + \lambda f = 0 \qquad f(a) = f(b) = 0$

a). $\lambda f = -(rf')' - pf$

Multiply f by each side, and integrate on $[a,b]$

$$\lambda \int_a^b |f|^2 dx = -\int_a^b (rf')' f \, dx - \int_a^b p|f|^2 dx$$

$$= -rff' \Big|_a^b + \int_a^b r|f'|^2 dx - \int_a^b p|f|^2 dx$$

$$= \int_a^b r|f'|^2 dx - \int_a^b p|f|^2 dx$$

b) $\lambda \int_a^b |f|^2 dx \geq \int_a^b r|f'|^2 dX - C \int_a^b |f|^2 dx$

$(\lambda + C) \int_a^b |f|^2 dx \geq \int_a^b r|f'|^2 dx \geq 0.$

$\lambda \geq -C$

c). $(rf')' + pf + \lambda f = 0. \qquad f'(a) - \alpha f(a) = f'(b) - \beta f(b) = 0$

Similarly $\lambda f = -(rf')' - pf$

$\lambda |f|^2 = -(rf')' f - p|f|^2.$

$$\lambda \int_a^b |f|^2 dx = -\int_a^b (rf')' f \, dx - \int_a^b p|f|^2 dx$$

$$\lambda \int_a^b |f|^2 dx = -rf'f \Big|_a^b + \int_a^b r|f'|^2 dx - \int_a^b p|f|^2 dx$$

$$\lambda \int_a^b |f|^2 dx = \int_a^b r|f'|^2 dx - \int_a^b p|f|^2 dX + r(a) f'(a) f(a) - r(b) f'(b) f(b)$$

Since $f'(a) = \alpha f(a) \qquad f'(b) = \beta f(b)$

$$\lambda \int_a^b |f|^2 dx = \int_a^b r|f'|^2 dx - \underline{\int_a^b p|f|^2 dX} + \alpha r(a)|f(a)|^2 - \beta r(b)|f(b)|^2$$

$\alpha \geq 0 \qquad \beta \leq 0.$

$$\lambda \int_a^b |f|^2 dX \geq -C \int_a^b |f|^2 dX + \int_a^b r|f|^2 dX + \alpha r(a)|f(a)|^2 - \beta r(b)|f(b)|^2$$

$\Rightarrow (\lambda + C) \int_a^b |f|^2 dx \geq 0.$

$\Rightarrow \lambda \geq -C.$