4.2.1. $u_t = k u_{xx}$, $u(0,t)=0$. $u_x(l,t)=0$ $u(x,0)=f(x)$.

Let $u(x,t) = X(x)\,T(t)$. Applying boundary conditions and intial condition,

$X(0)=0$. $X'(l)=0$ and $T(0)=f(x)$.

$X(x)\,T'(t) = X''(x)\,T(t)\cdot k$, $\dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{T(t)\cdot k}$ = constant.

i) constant $= \alpha^2 > 0$

$T'(t) = k\alpha^2\,T(t)$ and $X''(x) = \alpha^2 X(x)$.

$T(t) = A\,e^{k\alpha^2}$ $X(t) = A\cosh\alpha x + B\sinh\alpha x$

$X(0)=0 \Rightarrow A=0$

$X'(l) = \alpha B\cos(l\alpha) = 0 \Rightarrow B=0$ "

Trivial sol. "

ii) constant $=0$

$X(x) = Ax+B$. $X(0) = B = 0$

$X'(x) = A$ $X'(l) = A = 0$ $\Rightarrow$ Trivial sol.

iii) constant $= -\alpha^2 < 0$

$X(x) = A\cos\alpha x + B\sin\alpha x$

$X(0) = A = 0$ $X'(l) = \alpha B\cos l\alpha = 0$

$\alpha l = \dfrac{2n-1}{2}\pi$. $\therefore \alpha_n = \dfrac{2n-1}{2l}\pi$

$\therefore X(t) = B_n \sin\left(\dfrac{2n-1}{2l}\pi\right)x$

$T(t) = A_1\cdot e^{-k\left(\frac{2n-1}{2l}\pi\right)^2 t}$ "

$\therefore u_n(x,t) = C_n \sin\dfrac{2n-1}{2l}\pi x\cdot e^{-k\left(\frac{2n-1}{2l}\pi\right)^2 t}$.

But $u_n(x,0) = f(x)$, $\Rightarrow$ $C_n\sin\dfrac{2n-1}{2l}\pi x = f(x)$

$\displaystyle\int_0^l C_n \sin^2\frac{2n-1}{2l}\pi x\cdot dx = \int_0^l \sin\frac{2n-1}{2l}\pi x\cdot f(x)\,dx$.

$C_n\cdot\dfrac{l}{2} = \displaystyle\int_0^l f(x)\cdot\sin\frac{2n-1}{2l}\pi x\cdot dx$ $\therefore C_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{2n-1}{2l}\pi x\cdot dx$ "

For $f(x)=50$. , $C_n = \dfrac{2}{l}\displaystyle\int_0^l 50\cdot\sin\frac{2n-1}{2l}\pi x\cdot dx = \frac{100}{\pi l(2n-1)}\left[\frac{-\cos\left(\frac{(2n-1)\pi}{2l}\right)x}{\left(\frac{\ }{2l}\right)}\right]_0^l$

$= \begin{cases}\dfrac{200}{\pi l(2n-1)} & \text{for } n=\text{odd}\\ 0 & \text{for } n=\text{even}\end{cases}$

$\therefore u(x,t) = \displaystyle\sum^{\infty} \frac{200}{\pi l(2n-1)}\,e^{-k\left(\frac{2n-1}{2l}\pi\right)^2 t}\cdot\sin\left(\frac{2n-1}{2l}\right)\pi x$ "

4.2.2.

Solve. $\begin{cases} u_t = k u_{xx} \\ u(0,t) = C \neq 0 \quad u_x(l,t) = 0 \\ u(x,0) = 0 \end{cases}$

Let $v(x,t) = u(x,t) - C$. $v(x,t)$ satisfies.

$$\begin{cases} v_t = k v_{xx} \\ v(0,t) = 0 \quad v_x(l,t) = 0 \\ v(x,0) = f(x) - C. \end{cases}$$

We have solved this homogeneous problem in 4.2.1.

$$v(x,t) = \sum C_n \sin\left(\frac{2n-1}{2l}\pi x\right) e^{-k\left(\frac{2n-1}{2l}\pi\right)^2 t}.$$

Next, use the initial condition. $v(x,0) = f(x) - C$ to determine C_n

$$v(x,0) = \sum C_n \sin\left(\frac{2n-1}{2l}\pi x\right) = f(x) - C.$$

Since $\left\{ \sqrt{\frac{2}{l}} \sin\left(\frac{2n-1}{2l}\pi x\right) \right\}_{n=1}^{\infty}$ is an orthonormal set in $L^2(0,l)$

(By Exercise 3.3.6)

$$C_n = \frac{\left\langle f(x) - c,\ \sin\left(\frac{2n-1}{2l}\pi x\right) \right\rangle}{\left\| \sin\left(\frac{2n-1}{2l}\pi x\right) \right\|^2_{L^2(0,l)}} = \frac{\int_0^l (f(x)-c) \sin\left(\frac{2n-1}{2l}\pi x\right) dx}{\sqrt{\frac{l}{2}}\sqrt{\frac{l}{2}}}$$

$$= \frac{2}{l} \int_0^l [f(x) - c] \sin\left(\frac{2n-1}{2l}\pi x\right) dx$$