4.23. sol.) Let $u(x,t) = w(x,t) + Ax$

Then, $u(x,t)$ satisfies $u(0,t)=0$, $u_x(l,t)=A$, $u(x,0)=f(x)$ as well as the heat equation $u_t = ku_{xx}$

This is equivalent to $w_t = kw_{xx}$, $w(0,t) = w_x(l,t)=0$, $w(x,0)= f(x)-Ax$.

From 4.2.1. a),

$$w(x,t) = \sum_{n=1}^{\infty} d_n \exp \frac{-(n-\frac{1}{2})^2\pi^2 kt}{l^2} \sin (n-\tfrac{1}{2}) \frac{\pi x}{l}, \quad \text{where}$$

$$d_n = \frac{2}{l}\int_0^l (f(x) - Ax) \sin \frac{(2n-1)\pi x}{2l}\, dx$$

$$\frac{2}{l}\int_0^l Ax \sin \frac{(2n-1)\pi}{2l}x \, dx = \frac{2A}{l}\left[\frac{4l^2}{(2n-1)^2\pi^2}(-1)^{n-1}\right].$$

and let $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{(2n-1)\pi x}{2l}\, dx$

Then $w(x,t) = \sum_{n=1}^{\infty} \left(b_n - \frac{2A}{l}\left[\frac{4l^2}{(2n-1)^2\pi^2}(-1)^{n-1}\right]\right) \exp\frac{-(n-\frac{1}{2})^2\pi^2kt}{l^2} \sin (n-\tfrac{1}{2})\frac{\pi x}{l}.$

$\therefore u(x,t) = Ax + \sum_{n=1}^{\infty}\left(b_n + \frac{8Al(-1)^n}{(2n-1)^2\pi^2}\right) \exp\frac{-(n-\frac{1}{2})^2\pi^2kt}{l^2} \sin(n-\tfrac{1}{2})\frac{\pi x}{l}$ //

4.25. $u_t = ku_{xx} + e^{-2t}\sin x$ $u(x,0) = u(0,t) = u(\pi,t) = 0$. ; Inhomogeneous heat eq.

Let $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin nx$ and $e^{-2t}\sin x = \sum_{n=1}^{\infty}\beta_n(t)\sin nx$ //

Then $\beta_n(t) = \begin{cases} e^{-2t}, & n=1 \\ 0, & \text{otherwise} \end{cases}$

$u_t(x,t) = \sum_{n=1}^{\infty} b_n'(t)\sin nx$ $ku_{xx} = -k\sum_{n=1}^{\infty} n^2 b_n(t)\sin nx$

Therefore, given equation should be

$$\sum_{n=1}^{\infty} b_n'(t)\sin nx + k\sum_{n=1}^{\infty} n^2 b_n(t)\sin nx = \sum_{n=1}^{\infty}\beta_n(t)\sin nx$$

$\Rightarrow b_n'(t) + n^2k\, b_n(t) = \beta_n(t)$

For $n \neq 1$, $\beta_n(t)=0$, so $b_n(t) = ce^{-n^2kt}$, but $u(x,0)=0$.

$\Rightarrow c=0$.

For $n=1$, $b_1(t) = e^{-kt}\int_0^t e^{-2s}\cdot e^{ks}\, ds$

$= e^{-kt}\int_0^t e^{(k-2)s}\, ds$

If $k=2$, $b_1(t) = t\cdot e^{-2t}$ $\therefore u(x,t) = te^{-2t}\sin x$

If $k \neq 2$, $b_1(t) = e^{-kt}\int_0^t e^{(k-2)s}\, ds$

$= \frac{e^{-kt}}{k-2}[e^{kt}e^{-2t} - 1] = \frac{e^{-2t} - e^{-kt}}{k-2}$

$\therefore u(x,t) = \frac{e^{-2t}-e^{-kt}}{k-2}\sin x$