4.2.7

a) If we apply Technique 3. Let $u(x,t) = u_0(x,t) + V(x,t)$

where $\begin{cases} (\partial_t - k\partial_{xx})\, u_0(x) = R \\ u_{0x}(0) = 0 \quad u_{0x}(l) = 0. \end{cases}$ $\quad \begin{cases} V_t = kV_{xx} \\ V_x(0,t) = V_x(l,t) = 0 \\ V(x,0) = 0. \end{cases}$

$$\Downarrow$$

$$\begin{cases} -k u_{0xx} = R \\ u_{0x}(0) = u_{0x}(l) = 0. \end{cases}$$

$$u_0 = -\frac{k}{2R}x^2 + c_1 x + c_2$$

$$u_{0x} = -\frac{k}{R}x + c_1$$

$u_{0x}(0) = c_1 = 0 \qquad u_{0x}(l) = -\frac{k}{R}l + c_1 = 0$

$$\Downarrow$$

Get $\qquad c_1 = \frac{k}{R}l$

$\begin{cases} c_1 = 0 \\ c_1 = \frac{k}{R}l \end{cases}$ Contradiction! Therefore, technique 3 does not work here.

Physical Reason:

R represents a source in the rod. Two boundaries are insulated, so the temperature in the rod will definitely go up or down. Hence, the steady-state solution does not exist.

b) Solve it by technique 2.

Assume $u(x,t) = \sum w_n(t)\phi_n(x)$.

where $\phi_n(x)$ are eigenvectors of the Sturm-Liouville problem.

$$\begin{cases} \phi_n'' + \lambda_n \phi_n = 0 \\ \phi_{nx}(0) = \phi_{nx}(l) = 0 \end{cases} \Rightarrow \begin{cases} \phi_n(x) = \cos\frac{n\pi}{l}x \\ \lambda_n = \left(\frac{n\pi}{l}\right)^2 \end{cases}$$

$$u(x,t) = \sum w_n(t)\cos\frac{n\pi}{l}x$$

$$R = \sum a_n(t)\cos\frac{n\pi}{l}x \quad \text{where } \begin{cases} a_0(t) = R \\ a_n(t) = 0 \quad \text{where } n \neq 0 \end{cases}$$

Plug $u(x,t) = \sum w_n(x)\cos\frac{n\pi}{l}x$ into $u_t = ku_{xx} + R$.

$$\sum w_n'(t)\cos\frac{n\pi}{l}x = \sum -k\, w_n(t)\cdot\left(\frac{n\pi}{l}\right)^2\cos\frac{n\pi}{l}x + \sum a_n(t)\cos\frac{n\pi}{l}x$$

$$w_n'(t) + k\left(\frac{n\pi}{l}\right)^2 w_n(t) - a_n(t) = 0$$

solve this ODE

$$\begin{cases} w_0'(t) = a_0(t) = R \\ w_n'(t) = -k\left(\frac{n\pi}{\ell}\right)^2 w_n(t) & \text{where } n \neq 0 \end{cases}$$

get $w_0(t) = Rt + C_0$ $\quad w_n(t) = C_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t}$ when $n \neq 0$

Plug $w_0(t)$ & $w_n(t)$ to the solution

$$u(x,t) = \sum w_n(t) \cos\frac{n\pi}{\ell}x$$

$$= Rt + C_0 + \sum_{n\neq 0} C_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \cos\frac{n\pi}{\ell}x$$

$u(x,t)$ should satisfy the initial condition $u(x,0) = 0$

and boundary condition $u_x(0,t) = u_x(\ell,t) = 0$

therefore, $C_0 = C_n = 0$.

$$u(x,t) = Rt$$

How to make a smart guess?

$$\begin{cases} u_t = k u_{xx} + R \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = 0 \end{cases}$$

R is constant, and all initial and boundary condition are independent of x, we can assume $u_{xx} = 0$ and solve

$$\begin{cases} u_t = R \\ u\big|_{t=0} = 0 \end{cases}$$

$$u = \int_0^t R\, dt' = Rt$$

$\boxed{\text{part c}}$

Similarly, ϕ_n are eigenvectors in part (b).

Assume $u(x,t) = \sum w_n(t) \cos\frac{n\pi}{\ell}x$.

Plug $u(x,t)$ to the differential equation $u_t = k u_{xx} + Re^{-ct}$.
and get.

$$\sum w_n'(t) \cos\frac{n\pi}{\ell}x = \sum -k\, w_n(t) \left(\frac{n\pi}{\ell}\right)^2 \cos\frac{n\pi}{\ell}x + \sum a_n \cos\frac{n\pi}{\ell}x$$

where $\begin{cases} a_0(t) = Re^{-ct}. \\ a_n(t) = 0 \quad \text{when } n \neq 0. \end{cases}$

Solve ODE $\begin{cases} w_0'(t) = Re^{-ct}. \\ w_n'(t) = -k\left(\frac{n\pi}{\ell}\right)^2 w_n(t) \quad n \neq 0 \end{cases}$

$$\Rightarrow \begin{cases} w_0(t) = C_0 - \frac{R}{c}e^{-ct}. \\ w_n(t) = C_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \qquad n \neq 0. \end{cases}$$

Solution $\quad u(x,t) = C_0 - \frac{R}{c}e^{-ct} + \sum_{n\neq 0} C_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \cos\frac{n\pi}{\ell}x$

$u(x,t)$ should satisfies the initial condition $u(x,0) = 0$

$$u(x,0) = C_0 - \frac{R}{c} + \sum_{n\neq 0} C_n \cos\frac{n\pi}{\ell}x = 0$$

$$\Rightarrow C_0 = \frac{R}{c} \qquad C_n \neq 0 \text{ when } n \neq 0.$$

therefore, we get solution $u(x,t) = \frac{R}{c}\left(1 - e^{-ct}\right)$

How to make a smart guess?

Similarly, Let $u_{xx} = 0$. solve $\begin{cases} u_t = Re^{-ct} \\ u|_{t=0} = 0. \end{cases}$

$$\Rightarrow u(x,t) = \int_0^t Re^{-ct'}\,dt' = \frac{R}{c}\left(1 - e^{-ct}\right).$$

4.2.9. $u_t = (k u_x)_x + f(x,t)$, $u(0,t) = u(l,t) = u(x,0) = 0$

Let. $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\varphi_n(t)$ where $\varphi_n(t)$'s are eigenfunctions of $(kf')' + \lambda f = 0$ $f(0) = f(l) = 0$

and $f(x,t) = \sum_{n=1}^{\infty} \beta_n(t)\,\varphi_n(t)$

Then, $u_t - (k \cdot u_x)_x = f(x,t)$

$$\Rightarrow \sum_{n=1}^{\infty} b_n'(t)\,\varphi_n(t) - k \sum_{n=1}^{\infty} b_n(t)\,\varphi_n''(x) - k'(x) \sum_{n=1}^{\infty} b_n(t)\,\varphi_n'(x) = \sum_{n=1}^{\infty} \beta_n(t) \cdot \varphi_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n'(t)\,\varphi_n(x) + \sum_{n=1}^{\infty} \lambda_n \cdot b_n(t) \cdot \varphi_n(x) = \sum_{n=1}^{\infty} \beta_n(t)\,\varphi_n(x)$$

$(\because kf'' + k'f' = -\lambda f.)$

$$\Rightarrow b_n'(t) + \lambda_n b_n(t) = \beta_n(t)$$

$$\Rightarrow b_n(t) = e^{-\lambda_n t} \int_0^t \beta_n(t) \cdot e^{\lambda_n s}\, ds$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\, e^{-\lambda_n t} \int_0^t \beta_n(s)\, e^{\lambda_n s}\, ds. \quad //$$