4.3.3. $u_{tt} = c^2 u_{xx} - g$. $u(0,t) = u(l,t) = 0$. $u(x,0) = f(x)$. $u_t(x,0) = g(x)$

(a) Since φ is time-independent, $\varphi = \varphi(x)$.

It follows that $c^2 \varphi_{xx} = g$. For $c \neq 0$, $\varphi(x) = \frac{g}{2c^2} x^2 + ax + b$.

Since $\varphi(0) = 0 = \varphi(l) = 0$, $\varphi(x) = \frac{g x(x-l)}{2c^2}$ //

(b). Let $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{l}$,

$$F(x,t) = -g = \sum_{n=1}^{\infty} \beta_n(t) \sin \frac{n\pi x}{l}.$$

$$\beta_n(t) = \frac{2}{l}\int_0^l (-g) \sin \frac{n\pi x}{l} dx = \frac{-2g}{n\pi}(1-(-1)^n) \text{ . //}$$

$$b_n(t) = \frac{l}{n\pi c}\int_0^t \sin \frac{n\pi c(t-s)}{l} \cdot \beta_n(s) \cdot ds \qquad (4.21.)$$

$$= \frac{-4lg}{n^2\pi^2 c}\int_0^t \sin \frac{n\pi c(t-s)}{l} \cdot ds \qquad (n = \text{odd})$$

$$= \frac{4l^2 g}{n^3\pi^3 c^2}\left(\cos \frac{n\pi ct}{l} - 1\right)$$

$$\therefore \; u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_{2n-1} \sin \frac{(2n-1)\pi x}{l}$$

$$= \sum_{n=1}^{\infty} \frac{4l^2 g}{(2n-1)^3\pi^3 c^2} \cos \frac{(2n-1)\pi ct}{l} \cdot \sin \frac{(2n-1)\pi x}{l} - \sum_{n=1}^{\infty} \frac{4l^2 g}{(2n-1)^3\pi^3 c^2} \cdot \sin \frac{(2n-1)\pi}{l} x$$

Let $\varphi(x) = \frac{g x(x-l)}{2c^2} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$, where $A_n = \frac{2}{l}\int_0^l \frac{g}{2c^2} x(x-l) \cdot \sin \frac{n\pi x}{l} dx$

Here. $A_n = \begin{cases} \frac{-4gl^2}{n^3\pi^3 c^2} & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even.} \end{cases}$

$$\therefore \; u(x,t) = \frac{g x(x-l)}{2c^2} + \frac{4l^2 g}{\pi^3 c^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)}{l}\pi x \cdot \cos \frac{(2n-1)\pi ct}{l}.$$

Since $\sin \frac{(2n-1)}{l}\pi x \cdot \cos \frac{(2n-1)}{l}\pi ct = \frac{1}{2}\left(\sin \frac{(2n-1)}{l}\pi(x+ct) + \sin \frac{2n-1}{l}\pi(x-ct)\right)$,

$u(x,t) = \varphi(x) + \frac{1}{2}\sum_{n=1}^{\infty} \alpha_n \left(\sin \frac{2n-1}{l}\pi(x+ct) + \sin \frac{2n-1}{l}\pi(x-ct)\right)$. where $\alpha_n = \frac{4l^2 g}{\pi^3 c^2} \cdot \frac{1}{(2n-1)^3}$.

Two sums on the right side are just the Fourier sine series for Φ, the odd $2l$-periodic extension of φ, evaluated at $x \pm ct$.

4.3.6

Solve by seperation of variables. $\begin{cases} u_{tt} + 2ku_t = c^2 u_{xx} \\ u(0,t) = u(l,t) = 0 \end{cases}$

Assume $u = T(t)X(x)$.

$$T''X + 2kT'X = c^2TX'' \Rightarrow \frac{T''+2kT'}{c^2T} = \frac{X''}{X} = \lambda.$$

solve $\begin{cases} X'' = \lambda X. \\ X(0) = X(l) = 0. \end{cases}$

When $\lambda \geq 0$ only trivial solutions exist.

when $\lambda < 0$ Assume $\lambda = -\nu^2$. $X = A\cos\nu x + B\sin\nu x$.

$X(0) = A = 0$

$X(l) = B\sin l\nu = 0 \Rightarrow \nu l = n\pi \quad \nu_n = \frac{n\pi}{l}$.

$X_n = \sin\frac{n\pi}{l}x$.

solve $T_n'' + 2kT_n' - \lambda_n c^2 T_n = 0$.

the corresponding algebraic equation is $t^2 + 2kt - c^2\lambda_n = 0$.

$$t = -k \pm \sqrt{k^2 - (\tfrac{n\pi}{l}c)^2}. \qquad t^2 + 2kt + (\tfrac{n\pi}{l}c)^2 = 0.$$

Case1

If $k < \frac{n\pi}{l}c$. $t = -k \pm i\sqrt{(\frac{n\pi}{l}c)^2 - k^2}$. $T_n(t) = e^{-kt}\left(a_n\cos\sqrt{(\frac{n\pi}{l})^2 - k^2}\,t + b_n\sin\sqrt{(\frac{n\pi}{l})^2 - k^2}\,t\right)$

Case2.

If $k \geq \frac{n\pi}{l}c$. $t = -k \pm \sqrt{k^2 - (\frac{n\pi}{l}c)^2}$ $T_n(t) = a_n e^{[-k+\sqrt{k^2-(\frac{n\pi}{l}c)^2}]t} + b_n e^{[-k-\sqrt{k^2-(\frac{n\pi}{l}c)^2}\,t]}$

In our problem $k < \frac{\pi c}{l} \leq \frac{n\pi}{l}c$. (for all $n \geq 1$), so our solution of $T_n(t)$ are like case 1.

$$u(x,t) = \sum_{n\geq 1} T_n(t)X_n(x).$$

$$= \sum_{n\geq 1} e^{-kt}\left(a_n \cos \mu_n t + b_n \sin \mu_n t\right) \sin \frac{n\pi}{\ell} x$$

where $\mu_n = \sqrt{\left(\frac{n\pi}{\ell}\right)^2 - k^2}$.

If $k > \frac{\pi c}{\ell}$, For some small n, solution of $T_n(t)$ are in Case 2,

For big n, solutions of $T_n(t)$ are in Case 1.

$$u(x,t) = \sum_{\text{small } n} e^{-kt}\left[a_n e^{\sqrt{k^2-(\frac{n\pi}{\ell}c)^2}\,t} + b_n e^{-\sqrt{k^2-(\frac{n\pi}{\ell}c)^2}\,t}\right] \sin \frac{n\pi}{\ell} x$$

$$+ \sum_{\text{big } n} e^{-kt}\left[a_n \cos \sqrt{\left(\frac{n\pi}{\ell}\right)^2 - k^2}\,t + b_n \sin \sqrt{\left(\frac{n\pi}{\ell}\right)^2 - k^2}\,t\right] \sin \frac{n\pi}{\ell} x$$