4.4.1 $\nabla^2 u = 0$ in D, $D: \{(x,y) : 0 \le x \le l,\ 0 \le y \le l\}$

$u(x,0) = u(0,y) = u(l,y) = 0$, $u(x,l) = x(l-x)$

sol) Let $u(x,y) = X(x)Y(y)$

Then $X''(x)Y(y) = X(x)Y''(y)$.

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} := \nu^2.$$

$$\Rightarrow X''(x) + \nu^2 X(x) = 0 \quad ; \quad X(0) = X(l) = 0 \quad \cdots ①$$

$$Y''(y) - \nu^2 Y(y) = 0 \quad ; \quad Y(0),\ Y(l) = x(l-x) \quad \cdots ②$$

① gives $X_n(x) = A_n \sin\frac{n\pi x}{l}$

② gives $Y_n(y) = B_n \sinh\frac{n\pi y}{l}$.

Thus, $u_n(x,y) = \alpha_n \sin\frac{n\pi x}{l} \cdot \sinh\frac{n\pi y}{l}$

Expanding $x(l-x)$ into Fourier sine series,

$$x(l-x) = \sum \alpha_n \sin\frac{n\pi x}{l}$$

$$\alpha_n = \frac{2}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\left(\frac{2l^3}{\pi^3}\right)\frac{(-1+\cos n\pi)}{n^3} = \begin{cases} \frac{8l^2}{\pi^3}\cdot\frac{1}{n^3} & n: \text{odd} \\ 0 & n: \text{even} \end{cases}$$

$$\therefore x(l-x) = \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}$$

Applying $u(l,y) = 0$,

$$u_n(x,l) = \sum_{n=1}^{\infty}\alpha_{2n-1}\sin\frac{(2n-1)\pi x}{l}\cdot\sinh(2n-1)\pi = \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}$$

$$\Rightarrow \alpha_{2n-1} = \frac{8l^2}{\pi^3}\cdot\frac{1}{(2n-1)^3\cdot\sinh(2n-1)\pi}$$

$$\therefore u(x,y) = \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3\sinh(2n-1)\pi}\sinh\frac{(2n-1)\pi y}{l}\cdot\sin\frac{(2n-1)\pi x}{l}\ //$$

4.4.3 $\nabla^2 u = 0$ in D, $D: \{(x,y) : 0 \le x \le l,\ 0 \le y \le l\}$.

$u_x(0,y) = u_x(l,y) = u_y(x,0) = 0$, $u_y(x,l) = f(x)$

sol) In the same way in 4.4.1.

$X''(x) + \nu^2 X(x) = 0$, $X'(0) = X'(l) = 0$

$\Rightarrow X_n(x) = A_n \cos\frac{n\pi x}{l}$

$Y''(y) - \nu^2 Y(y) = 0$, $Y'(0) = 0$

$\Rightarrow Y_n(y) = B_n \cosh\frac{n\pi y}{l}$

$$u(x,y) = \sum_{n=0}^{\infty}\alpha_n\cos\frac{n\pi x}{l}\cosh\frac{n\pi y}{l} = C + \sum_{n=1}^{\infty}\alpha_n\cos\frac{n\pi x}{l}\cdot\cosh\frac{n\pi y}{l}.$$

Expanding $f(x)$ into Fourier cosine series.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}.$$

$$u_y(x,l) = \sum_{n=1}^{\infty} \frac{\alpha_n n\pi}{l} \sinh n\pi \cdot \cos\frac{n\pi x}{l} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}.$$

$$\Rightarrow a_0 = 0, \qquad a_n = \frac{\alpha_n n\pi}{l} \sinh n\pi$$

$$\therefore u(x,y) = C + \sum_{n=1}^{\infty} \frac{l a_n}{n\pi \cdot \sinh n\pi} \cos\frac{n\pi x}{l} \cdot \cosh\frac{n\pi y}{l}.$$