4.45. $\nabla^2 u = 0$. $(r_0 \le r \le 1)$

a. The general solution is given by $u(r,\theta) = a_0 + b_0 \log r + \sum_{n=\pm1,\dots} e^{in\theta}(a_n r^n + b_n r^{-n})$ (4.33)

$$u_r(r_0,\theta) = \frac{b_0}{r_0} + \sum_{n=\pm1,\pm2,\dots} e^{in\theta}(n a_n r_0^{n-1} - n b_n r_0^{-(n+1)}) = 0$$

$$\Rightarrow b_0 = 0 \quad \text{and} \quad a_n r_0^{n-1} = b_n r_0^{-(n+1)}$$

$$a_n r_0^{2n} = b_n$$

$$u(r,\theta) = a_0 + \sum_{n=\pm1,\pm2,\dots} e^{in\theta}(a_n r^n + a_n r_0^{2n} \cdot r^{-n})$$

Since $u(1,\theta) = f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}$, $u(1,\theta) = a_0 + \sum_{n=\pm1,\pm2,\dots} e^{in\theta}(a_n + b_n)$

$$= \sum_{-\infty}^{\infty} c_n e^{in\theta}$$

Therefore, $c_n = a_n + b_n \quad (n \ne 0) \quad (\Rightarrow c_n = a_n + a_n r_0^{2n} = a_n(1 + r_0^{2n}))$

$c_0 = a_0$

$$u(r,\theta) = c_0 + \sum_{n=\pm1,\pm2,\dots}\left(\frac{c_n}{1+r_0^{2n}} r^n + \frac{c_n}{1+r_0^{2n}} \cdot r_0^{2n} r^{-n}\right)$$

$$= c_0 + \sum_{n=\pm1,\dots}\left(\frac{r^n + r_0^{2n} r^{-n}}{1+r_0^{2n}}\right) c_n e^{in\theta} = \sum_{-\infty}^{\infty} c_n \left(\frac{r^n + r_0^{2n} r^{-n}}{1+r_0^{2n}}\right) \cdot e^{in\theta}$$

4.4.7

Solve $\nabla^2 u = 0$

$$u(r_0,\theta) = u(1,\theta) = 0 \qquad u(r,0) = g(r) \quad u(r,\beta) = h(r)$$

Use polar coordinate, Assume $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda.$$

Get ODE $\quad r^2R''(r) + rR'(r) + \lambda R(r) = 0 \qquad ①$

$$\begin{cases} r^2R''(r) + rR'(r) + \lambda R(r) = 0 \\ R(1) = R(r_0) = 0 \end{cases}$$

This is a second order Euler equation, Let. $z = \log r$.

$$\frac{dR}{dr} = \frac{dR}{dz}\frac{dz}{dr} = \frac{dR}{dz}\frac{1}{r}.$$

$$\frac{d^2R}{dr^2} = \frac{d^2R}{dz^2}\frac{dz}{dr}\frac{1}{r} + \frac{dR}{dz}\left(-\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{d^2R}{dz^2} - \frac{1}{r^2}\frac{dR}{dz}$$

Plug in to ① and get another ODE about $R(z)$

$$r^2\left(\frac{1}{r^2}\frac{d^2R}{dz^2} - \frac{1}{r^2}\frac{dR}{dz}\right) + r\frac{1}{r}\frac{dR}{dz} + \lambda R = 0.$$

$$\begin{cases} \dfrac{d^2R}{dz^2} + \lambda R(z) = 0. & ② \\ R\big|_{z=\log r_0} = R\big|_{z=\log 1=0} = 0. \end{cases}$$

We can get a nontrivial solution if $\lambda > 0$. Let $\lambda = \nu^2$.

$$R(z) = A\cos\nu z + B\sin\nu z.$$

$$R(0) = A = 0.$$

$$R(\log r_0) = B\sin\nu\log r_0 = 0. \Rightarrow \nu = \frac{n\pi}{\log r_0}.$$

Get solution of ②: $\quad R_n(z) = B_n\sin\dfrac{n\pi z}{\log r_0} \qquad \lambda_n = \left(\dfrac{n\pi}{\log r_0}\right)^2.$

Plug in $z = \log r$. $R_n(r) = B_n \sin \frac{n\pi \log r}{\log r_0}$ $\lambda_n = \left(\frac{n\pi}{\log r_0}\right)^2$.

On the other hand, solve for $\bar{\Theta}(\theta)$

$$\bar{\Theta}(\theta) = \lambda \bar{\Theta}(\theta)$$

Since $\lambda_n = \left(\frac{n\pi}{\log r_0}\right)^2$. $\bar{\Theta}(\theta) = a_n e^{\frac{n\pi\theta}{\log r_0}} + b_n e^{-\frac{n\pi\theta}{\log r_0}}$.

Finally, $u(r,\theta) = \sum \left(a_n e^{\frac{n\pi\theta}{\log r_0}} + b_n e^{-\frac{n\pi\theta}{\log r_0}}\right) \sin \frac{n\pi \log r}{\log r_0}$

where $u(r,0) = \sum (a_n + b_n) \sin \frac{n\pi \log r}{\log r_0} = g(r)$

$$u(r,\beta) = \sum \left(a_n e^{\frac{n\pi\beta}{\log r_0}} + b_n e^{-\frac{n\pi\beta}{\log r_0}}\right) \sin \frac{n\pi \log r}{\log r_0} = h(r).$$