1.2.5 a. $u_{n_x}(x,y) = n\pi \cos(n\pi x) \cdot \sinh(n\pi y)$

$u_{n_{xx}}(x,y) = -n^2\pi^2 \sin(n\pi x) \sinh(n\pi y)$

$u_{n_y}(x,y) = n\pi \sin(n\pi x) \cosh(n\pi y)$

$u_{n_{yy}}(x,y) = n^2\pi^2 \sin(n\pi x) \sinh(n\pi y)$

$\Rightarrow u_{n_{xx}}(x,y) + u_{n_{yy}}(x,y) = 0$

$u_n(0,y) = 0 \cdot \sinh(n\pi y) = 0$

$u_n(1,y) = \sin(n\pi) \cdot \sinh(n\pi y) = 0$

$u_n(x,0) = \sin(n\pi x) \cdot \sinh(0) = 0.$

b. For general solution $\sum_n a_n \sin(n\pi x) \cdot \sinh(n\pi y)$ to satisfy

$u(x,1) = \sin 2\pi x - \sin 3\pi x$, let

$\sum_n a_n \sin(n\pi x) \cdot \sinh(n\pi) = \sin 2\pi x - \sin 3\pi x$

Hence, $n=2$ gives $a_2 = \frac{1}{\sinh(2\pi)}$, $n=3$ gives $a_3 = -\frac{1}{\sinh(3\pi)}$

$a_n = 0$ for $n \neq 2, 3$.

$\therefore \frac{1}{\sinh(2\pi)} \sin(2\pi x) \cdot \sinh(2\pi y) - \frac{1}{\sinh(3\pi)} \sin(3\pi x) \cdot \sinh(3\pi y)$

c. $\tilde{u}_{n_x}(x,y) = n\pi \cdot \cos(n\pi x) \sinh n\pi(1-y)$

$\tilde{u}_{n_{xx}}(x,y) = -n^2\pi^2 \sin(n\pi x) \sinh n\pi(1-y)$

$\tilde{u}_{n_y}(x,y) = -n\pi \sin(n\pi x) \cdot \cosh n\pi(1-y)$

$\tilde{u}_{n_{yy}}(x,y) = n^2\pi^2 \sin(n\pi x) \cdot \sinh n\pi(1-y)$.

$\Rightarrow \tilde{u}_{n_{xx}}(x,y) + \tilde{u}_{n_{yy}}(x,y) = 0.$

$\tilde{u}_n(0,y) = \sin(0) \cdot \sinh n\pi(1-y) = 0$

$\tilde{u}_n(1,y) = \sin(n\pi) \cdot \sinh n\pi(1-y) = 0$

$\tilde{u}_n(x,1) = \sin(n\pi x) \cdot \sinh(0) = 0$

d. For $\sum_n a_n \sin(n\pi x) \cdot \sinh n\pi(1-y)$ to satisfy

$u(x,0) = 2\sin\pi x$, let

$\sum_n a_n \sin(n\pi x) \cdot \sinh n\pi = 2\sin\pi x$.

Hence, $a_n = 0$ for $n \neq 1$ and $a_1 \sin\pi x \cdot \sinh n\pi = 2\sin\pi x$

$\therefore a_1 = \frac{2}{\sinh\pi}$

$\therefore \frac{2}{\sinh\pi} \cdot \sin(\pi x) \cdot \sinh \pi(1-y)$

e. From the given Dirichlet problem, consider the following two equations with b.v. conditions.

① $u_{xx} + u_{yy} = 0$ with $u(0,y) = u(1,y) = 0$ and $u(x,0) = 2\sin \pi x$.

② $u_{xx} + u_{yy} = 0$ with $u(0,y) = u(1,y) = 0$ and $u(x,1) = \sin 2\pi x - \sin 3\pi x$.

From c. and d., the solution of ① is $\frac{2}{\sinh(\pi)} \sin(\pi x) \sinh \pi(1-y)$

From a. and b. the solution of ② is

$$\frac{\sin(2\pi x)\cdot \sinh(2\pi y)}{\sinh(2\pi)} - \frac{\sin(3\pi x)\cdot \sinh(3\pi y)}{\sinh(3\pi)}$$

Hence, by the Superposition principle, the solution is

$$u(x,y) = \frac{\sin(2\pi x)\sinh(2\pi y)}{\sinh(2\pi)} - \frac{\sin(3\pi x)\cdot\sinh(3\pi y)}{\sinh(3\pi)} + 2\frac{\sin(\pi x)\sinh \pi(1-y)}{\sinh(\pi)}$$ //

1.3.7. Let $u(x,t) = X(x)T(t)$.

Then, the given equation will be

$$X(x)T'(t) = k X''(x)T(t).$$

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} := -\lambda^2 = \alpha.$$

$X'' + \lambda^2 X = 0 \Rightarrow X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x.$

$T' + \lambda^2 k T = 0 \Rightarrow T(t) = c_3 e^{-\lambda^2 kt}$

Imposing b.c.,

$u(0,t) = c_1 T(t) = 0, \ \forall t \Rightarrow c_1 = 0.$

$u_x(l,t) = \lambda c_2 \cos \lambda l \ T(t) = 0 \ \forall t \Rightarrow \lambda l = \frac{2n+1}{2}\pi$

$\therefore \lambda = \frac{2n+1}{2l}\pi.$

$\therefore u(x,t) = C \sin \frac{2n+1}{2l}\pi x \, e^{-\frac{(2n+1)^2}{4l^2}\pi^2 kt}$ //

For $\alpha = 0$, we have a trivial solution

For $\alpha > 0$ ($\alpha = \lambda^2$),

$X(x) = c_1 \cosh \lambda x + c_2 \sinh \lambda x.$

$X(0) = c_1 \cosh \lambda x = 0 \Rightarrow c_1 = 0$

$X'(l) = c_2 \lambda \cosh \lambda x = 0 \Rightarrow c_2 = 0$ ($\because \cosh \lambda x > 0$)

$X(t) = 0$: trivial solution.