1.3.3. Given equation : $u_{tt} = 9u_{xx}$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = 2\sin \pi x - 3\sin 4\pi x \quad \cdots \text{①}$$

$$u_t(x,0) = 0 \quad (0 < x < 1) \quad \cdots \text{②}$$

Using the result in P.16, we have the series solution

$$u(x,t) = \sum_1^\infty \sin(n\pi x)\left(a_n \cos 3n\pi t + b_n \sin 3n\pi t\right)$$

Now, we impose initial conditions ① and ②.

By ①, $u(x,0) = \sum_{n=1}^\infty \sin(n\pi x)\left(a_n \cos(3n\pi\cdot 0) + b_n \sin(3n\pi \cdot 0)\right)$

$$= \sum_{n=1}^\infty \sin(n\pi x) a_n$$

$$= 2\sin \pi x - 3\sin 4\pi x.$$

$\Rightarrow$ $a_1 = 2.$ $a_4 = -4.$ and $a_n = 0$ for $n \neq 2, -4.$

$$u_t(x,t) = \sum_{n=1}^\infty \sin(n\pi x)\cdot\left(-a_n 3n\pi \cdot \sin 3n\pi t + b_n 3n\pi \cos 3n\pi t\right)$$

$$u_t(x,0) = \sum_{n=1}^\infty \sin(n\pi x)(b_n 3n\pi) = 0$$

$\Rightarrow$ $b_n = 0$ for all n.

$\therefore$ $u(x,t) = 2\sin\pi x \cdot \cos 3\pi t - 3\sin 4\pi x \cdot \cos 12\pi t.$

1.3.4.

Let $u(x,t) = X(x)T(t)$

$u_t = X(x)T'(t)$ $\quad$ $u_{xx} = X''(x)T(t)$

Plug into the equation $u_t = \frac{1}{10}u_{xx}$, $XT' = \frac{1}{10}X''T$

$\Rightarrow \frac{10T'}{T} = \frac{X''}{X} = A$. where A is a constant

Get two ODE $\begin{cases} X'' = AX & ① \\ T' = \frac{1}{10}AT & ② \end{cases}$

$u(x,t)$ satisfies boundary condition $u_x(0,t) = u_x(\pi,t) = 0$

$\Rightarrow X'(0) = X'(\pi) = 0$

Solve ① $\quad X(x) = C_1\cos\lambda x + C_2\sin\lambda x \qquad \lambda = \sqrt{-A}$

$X'(x) = -\lambda C_1\sin\lambda x + \cancel{\lambda C_2}\cos\lambda x$

$X'(0) = \lambda C_2 = 0 \quad \Rightarrow C_2 = 0$

$X'(\pi) = -\lambda C_1\sin\lambda\pi = 0 \Rightarrow \lambda = n$ where $n \in \mathbb{Z}$

$\Rightarrow X_n(x) = C_n\cos nx$

Given $\lambda = n$. $A = -\lambda^2 = -n^2$.

Solve ② $\quad T_n(t) = C_n e^{\frac{1}{10}At} = C_n e^{-\frac{n^2}{10}t}$.

General solution $u(x,t) = \sum_{n=-\infty}^{+\infty} a_n e^{-\frac{n^2}{10}t}\cos nx$

Plug the initial condition $u(x,0) = 3-4\cos 2x$ to the general solution.

$\sum_{n=-\infty}^{+\infty} a_n\cos nx = 3-4\cos 2x$

$\Rightarrow \begin{cases} a_0 = 3 \\ a_2 = -4 \\ a_n = 0 \quad \text{when } n \neq 0,2 \end{cases}$

$u(x,t) = 3 - 4e^{-\frac{2}{5}t}\cos 2x$

Let $|u(x,t)-3| < 10^{-4} \Rightarrow |4e^{-\frac{2}{5}t}\cos 2x| < 10^{-4}$

Since $|\cos 2x| < 1$, we need $4e^{-\frac{2}{5}t} < 10^{-4}$

$\Rightarrow t > -\frac{5}{2}\ln\frac{10^{-4}}{4} \qquad t_0 = -\frac{5}{2}\ln\frac{10^{-4}}{4}$