9. $f(\theta) = |\cos\theta|$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos\theta| \cdot d\theta = \frac{2}{\pi}\int_0^{\pi} |\cos\theta| \cdot d\theta$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos\theta \cdot d\theta = \frac{4}{\pi} \sin\theta \Big|_0^{\frac{\pi}{2}} = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos\theta| \cdot \cos n\theta \cdot d\theta$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos\theta \cdot \cos n\theta \cdot d\theta - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos\theta \cdot \cos n\theta \, d\theta.$$

Using the identity $\cos\theta_1 \cdot \cos\theta_2 = \frac{1}{2}\left(\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2)\right)$,

$$= \frac{2}{\pi}\left[\frac{n\cos\theta\sin n\theta - \cos\theta \sin n\theta}{n^2-1}\right]_0^{\frac{\pi}{2}} - \frac{2}{\pi}\left[\frac{n\cos\theta\sin n\theta - \cos n\theta \sin\theta}{n^2-1}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-(-1)^n}{4n^2-1}\right] - \frac{2}{\pi}\left[\frac{(-1)^n}{4n^2-1}\right] = -\frac{4}{\pi}\left[\frac{(-1)^n}{4n^2-1}\right].$$

$b_n = 0$. for all n since $f(\theta)$ is even, thus $f(\theta)\sin n\theta$ is odd.

$$\therefore f(\theta) = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^{\infty} \frac{(-1)^n \cos 2n\theta}{4n^2-1}$$

* a_n decays as fast as $\left(\frac{1}{n^2}\right)$; smoother than #4.

12. $f(\theta) = \begin{cases} (2a)^{-1} & |\theta| < a \\ 0 & a < |\theta| < \pi \end{cases}$

Since $f(\theta)$ is even, $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(\theta) d\theta = \frac{2}{\pi}\int_0^{a} \frac{1}{2a} d\theta = \frac{2}{\pi}\frac{a}{2a} = \frac{1}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta = \frac{2}{\pi}\int_0^{a} \frac{1}{2a}\cos n\theta \, d\theta = \frac{1}{\pi a}\int_0^{a} \cos n\theta d\theta$$

$$= \frac{1}{\pi a}\frac{1}{n}\sin n\theta \Big|_0^{a} = \frac{\sin na}{\pi a n}$$

Fourier coefficients

$$\frac{1}{2}a_0 + \sum_1^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

$$= \frac{1}{2\pi} + \frac{1}{\pi}\sum_1^{\infty} \frac{\sin na}{na}\cos n\theta$$

If we periodize $f(\theta)$, it's discontinuous, so Fourier coeffeients a_n decays like $O(\frac{1}{n})$, which is very slow.

17. $f(\theta) = \theta(\pi - |\theta|) \qquad (-\pi < \theta < \pi)$

$a_0 = a_n = 0$ since $f(\theta) = -f(-\theta)$, odd.

$$b_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cdot\sin n\theta \cdot d\theta = \frac{2}{\pi}\int_0^\pi (\theta\pi - \theta^2)\cdot \sin n\theta\, d\theta$$

$$= 2\int_0^\pi \theta\cdot\sin n\theta\cdot d\theta - \frac{2}{\pi}\int_0^\pi \theta^2\cdot \sin n\theta\, d\theta$$

$$= 2\left[-\frac{\theta\cdot\cos n\theta}{n} + \frac{\sin n\theta}{n^2}\right]_0^\pi - \frac{2}{\pi}\left[\frac{2\theta}{n^2}\sin n\theta + \frac{2}{n^3}\cos n\theta - \frac{\theta^2}{n}\cos n\theta\right]_0^\pi$$

integration by part.

$$= 2\left(-\frac{\pi}{n}(-1)^n\right) - \frac{2}{\pi}\left(\left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)(-1)^n - \frac{2}{n^3}\right)$$

$$= \frac{4}{\pi}\left(\frac{1-(-1)^n}{n^3}\right).$$

$$f(\theta) = \frac{4}{\pi}\sum \frac{1-(-1)^n}{n^3}\sin n\theta = \frac{8}{\pi}\sum \frac{1}{(2n-1)^3}\sin(2n-1)\theta.$$

(b_n) decays as fast as $\frac{1}{n^3}$: $f(\theta)$ is smooth

Conclusion: Function in 9 is continuous, so Fourier coefficients decay like O(1/n^2)

Function in 12 is discontinuous, so Fourier coefficients decay like O(1/n)

Both function and its derivative in 17 are continuous, so Fourier coefficients decay like O(1/n^3).

The smoother the function is, the faster its Fourier coefficients decay.