3.1.1. $\|a+b\|^2 = \langle a+b, \overline{a+b} \rangle = \langle a, \bar{a} \rangle + \langle a, \bar{b} \rangle + \langle b, \bar{a} \rangle + \langle b, \bar{b} \rangle$

$\|a-b\|^2 = \langle a-b, \overline{a-b} \rangle = \langle a, \bar{a} \rangle - \langle a, \bar{b} \rangle - \langle b, \bar{a} \rangle + \langle b, \bar{b} \rangle$

$\therefore \|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$ for all $a, b \in \mathbb{C}^n$.

* You can show the identity by using $\|x\|^2 = \sum_{i=1}^{n} x_i^2$.

HW 8.

2.6.6 (a) Show that $g_N'(\theta) = 2\pi \cdot D_N(\theta)$

$$g_N'(\theta) = \frac{d}{d\theta}\left(2 \cdot \sum_{n=1}^{N} \frac{\sin n\theta}{n} - \pi + \theta\right)$$

$$= 2 \cdot \sum_{n=1}^{N} \cos n\theta + 1. = 2\pi\left(\frac{1}{\pi} \cdot \sum_{n=1}^{N} \cos n\theta + \frac{1}{2\pi}\right)$$

$$= 2\pi \cdot D_N(\theta)$$

(b) Set $g_N'(\theta) = 0$, then

$$2\pi \cdot D_N(\theta) = 2\pi \cdot \frac{1}{2\pi} \cdot \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} = 0.$$

Note that $\sin(\frac{1}{2}\theta) \neq 0$ for $\theta \in (0, 2\pi)$

$\therefore \sin(N+\frac{1}{2})\theta = 0 \Rightarrow (N+\frac{1}{2})\theta = n\pi$

$\therefore$ the first critical pt. occurs at $\theta = \frac{\pi}{N+\frac{1}{2}}$.

$$g_N(\theta_N) - g_N(0) = \int_0^{\theta_N} g_N'(t)\, dt.$$

$$g_N(0) = 2 \cdot \sum_{n=1}^{N} \frac{\sin 0}{n} - (\pi - 0) = -\pi.$$

$$\Rightarrow g_N(\theta_N) + \pi = \int_0^{\theta_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\, d\theta.$$

$$\Rightarrow g_N(\theta_N) = \int_0^{\theta_N} \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\, d\theta - \pi.$$

(c). Let $\alpha = N+\frac{1}{2}\theta$., then $d\alpha = (N+\frac{1}{2})d\theta$.

$$g_N(\theta_N) = \int_0^{\pi} \frac{\sin\alpha\, d\alpha}{(N+\frac{1}{2})\sin\left(\frac{\alpha}{2N+1}\right)} - \pi = \int_0^{\pi} \frac{2 \cdot \left(\frac{\alpha}{2N+1}\right)}{\sin\left(\frac{\alpha}{2N+1}\right)} \cdot \frac{\sin\alpha}{\alpha}\, d\alpha - \pi.$$

Now, $\lim_{N\to\infty} g_N(\theta_N) = \lim_{N\to\infty} \int_0^{\pi} \frac{2\left(\frac{\alpha}{2N+1}\right)}{\sin\left(\frac{\alpha}{2N+1}\right)} \cdot \frac{\sin\alpha}{\alpha}\, d\alpha - \pi$

$$= 2\int_0^{\pi} \lim_{N\to\infty}\left(\frac{\frac{\alpha}{2N+1}}{\sin\frac{\alpha}{2N+1}}\right) \cdot \frac{\sin\alpha}{\alpha}\, d\alpha$$

It is possible to exchange the integral and limit. " Dominated convergence theorem"

$$= 2\int_0^{\pi} \frac{\sin\alpha}{\alpha}\, d\alpha - \pi.$$

$(\ast\ \lim_{x\to 0} \frac{\sin x}{x} = 1)$