Problem 1 (30 pts)

(a) (10 pts) Compute the Fourier series of \( f(\theta) = \theta^2 \) on \((-\pi, \pi)\), which is \(2\pi\) periodic.

**Answer:** Since this function is an even function over \((-\pi, \pi)\), the Fourier series becomes a Fourier cosine series. Thus, we only need the Fourier cosine coefficients \(a_n, n = 0,1,2,\ldots\). For \(a_n, n \geq 1\), we have:

\[
     \begin{align*}
     a_n &= \frac{2}{\pi} \int_{0}^{\pi} \theta^2 \cos n\theta \, d\theta \\
     &= \frac{2}{\pi} \left\{ \left[ \frac{\theta^2 \sin n\theta}{n} \right]_{0}^{\pi} - \frac{1}{n} \int_{0}^{\pi} 2\theta \sin n\theta \, d\theta \right\} \quad \text{(Integration by Parts)} \\
     &= -\frac{4}{n\pi} \int_{0}^{\pi} \theta \sin n\theta \, d\theta \\
     &= -\frac{4}{n\pi} \left\{ \left[ -\frac{\theta \cos n\theta}{n} \right]_{0}^{\pi} + \frac{1}{n} \int_{0}^{\pi} \cos n\theta \, d\theta \right\} \\
     &= -\frac{4}{n\pi} \cdot \frac{\pi(-1)^{n+1}}{n} \\
     &= 4 \cdot \frac{(-1)^n}{n^2}.
     \end{align*}
\]

Now, \(a_0\) can be computed as:

\[
     a_0 = \frac{2}{\pi} \int_{0}^{\pi} \theta^2 \, d\theta = \frac{2\pi^2}{3}.
\]

Hence we have

\[
    \theta^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta.
\]

(b) (10 pts) Prove

\[
    \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},
\]

using the result of Part (a).

**Answer:** Evaluate the Fourier series of \(f(\theta)\) of Part (a) at \(\theta = \pi\). (Another easy way is to evaluate it at \(\theta = 0\), but I omit the proof for \(\theta = 0\) case here). Since \(\theta = \pi\) is a point of continuity, we have

\[
     \begin{align*}
     f(\pi) &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi \\
     &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} \\
     &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}.
     \end{align*}
\]

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Hence, it is easy to derive

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \left( \pi^2 - \frac{\pi^2}{3} \right) = \frac{\pi^2}{6}.
\]

(c) (10 pts) Using the result of Part (a), compute the Fourier series of \( g(\theta) = \theta \) on \((-\pi, \pi)\), which is also \(2\pi\) periodic.

Hint: Use the derivative formula.

**Answer:** Because the \(2\pi\) periodic function \(\theta^2\) is *continuous and piecewise smooth* over \(\mathbb{R}\), we can differentiate the result of Part (a) as:

\[
2\theta \sim 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-\sin n\theta),
\]

from which we can easily derive:

\[
\theta \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta.
\]
Problem 2 (20 pts) Consider a function space, \( L^2[-1,1] \).

(a) (4 pts) Assuming each function in \( L^2[-1,1] \) is complex-valued, i.e., \( f : [-1,1] \to \mathbb{C} \), state the standard definition of the \( L^2 \)-norm and the inner product of this space.

Answer: Clearly, the standard \( L^2 \)-norm for a function \( f \in L^2[-1,1] \) is:

\[
\| f \|_2 \triangleq \left( \int_{-1}^{1} |f(x)|^2 \, dx \right)^{1/2},
\]

and the inner product of \( f, g \in L^2[-1,1] \) is defined as:

\[
\langle f, g \rangle \triangleq \int_{-1}^{1} f(x) \overline{g(x)} \, dx.
\]

(b) (4 pts) State the Cauchy-Schwarz inequality for this space.

Answer: The Cauchy-Schwarz inequality for \( L^2[-1,1] \) is, for any \( f, g \in L^2[-1,1] \),

\[
|\langle f, g \rangle| \leq \| f \|_2 \| g \|_2,
\]

where the equality holds if and only if \( f \) is proportional to \( g \) almost everywhere.

(c) (6 pts) The space \( L^2[-1,1] \) is known to be complete with respect to the \( L^2 \)-norm. State the definition of the completeness of this space.

Answer: Let \( \{ f_n \} \subset L^2[-1,1] \) be a Cauchy sequence in \( L^2[-1,1] \), i.e., \( \| f_m - f_n \|_2 \to 0 \) as \( m, n \to \infty \).

The completeness means that every Cauchy sequence in \( L^2[-1,1] \) is a convergent sequence, i.e., there exists \( f \in L^2[-1,1] \) such that \( \| f_n - f \|_2 \to 0 \) as \( n \to \infty \).
(d) (6 pts) Show an example of an orthonormal set in $L^2[-1,1]$, which is not complete (i.e., not an orthonormal basis for $L^2[-1,1]$).

Answer: Consider the following set of functions:

$$
\phi_n(x) = \begin{cases} 
0 & \text{if } -1 \leq x \leq 0, \\
\sqrt{2} \sin n\pi x & \text{if } 0 \leq x \leq 1.
\end{cases} \quad n = 1, 2, \ldots
$$

Then, this set of function is an orthonormal set because:

$$
\langle \phi_m, \phi_n \rangle = \int_{-1}^{1} \phi_m(x)\phi_n(x) \, dx
= \int_{0}^{1} 2 \sin m\pi x \sin n\pi x \, dx
= \begin{cases} 
\int_{0}^{1} (\cos(m-n)\pi x - \cos(m+n)\pi x) \, dx & \text{if } m \neq n; \\
\int_{0}^{1} (1 - \cos 2m\pi x) \, dx & \text{if } m = n
\end{cases}
= \begin{cases} 
\frac{\sin(m-n)\pi x}{(m-n)\pi} - \frac{\sin(m+n)\pi x}{(m+n)\pi} \bigg|_{0}^{1} & \text{if } m \neq n; \\
\left[ x - \frac{\sin 2m\pi x}{2m\pi} \right]_{0}^{1} & \text{if } m = n
\end{cases}
= \delta_{mn}.
$$

However, this set clearly cannot represent all the functions in $L^2[-1,1]$ since $\phi_n(x) = 0$ for $n = 0, 1, \ldots$, i.e., $\{\phi_n\}_{n=1}^{\infty}$ cannot satisfy Parseval’s equality for functions in $L^2[-1,1]$. Therefore, $\{\phi_n\}_{n=1}^{\infty}$ is not a complete orthonormal set in $L^2[-1,1]$. 

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Problem 3 (25 pts) The first three Legendre polynomials are:

\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1). \]

(a) (10 pts) Show that they are mutually orthogonal. Moreover, compute their orthonormal version, \( \phi_0(x), \phi_1(x), \phi_2(x) \).

**Answer:** They are mutually orthogonal because:

\[
\langle P_0, P_1 \rangle = \int_{-1}^{1} 1 \cdot x \, dx = 0 \quad \text{because } x \text{ is an odd function.}
\]

\[
\langle P_0, P_2 \rangle = \int_{-1}^{1} 1 \cdot \frac{1}{2}(3x^2 - 1) \, dx = \frac{1}{2} \left[ x^3 - x \right]_{-1}^{1} = 0
\]

\[
\langle P_1, P_2 \rangle = \int_{-1}^{1} x \cdot \frac{1}{2}(3x^2 - 1) \, dx = 0 \quad \text{because } x^3 \text{ and } x \text{ are odd functions.}
\]

Now, let’s compute the \( L^2 \)-norm of them.

\[
\| P_0 \|_2 = \sqrt{\int_{-1}^{1} 1^2 \, dx} = \sqrt{2}
\]

\[
\| P_1 \|_2 = \sqrt{\int_{-1}^{1} x^2 \, dx} = \sqrt{\frac{2}{3}}
\]

\[
\| P_2 \|_2 = \sqrt{\int_{-1}^{1} \frac{1}{4}(3x^2 - 1)^2 \, dx} = \frac{1}{2} \int_{0}^{1} (9x^4 - 6x^2 + 1) \, dx = \frac{1}{2} \left( \frac{9}{5} - 2 + 1 \right) = \sqrt{\frac{2}{5}}.
\]

Note that one can also use the formula \( \| P_n \|_2 = \sqrt{\frac{2}{2n+1}}, \ n = 0, 1, 2, \ldots \) Therefore, we have:

\[
\phi_0 = \frac{P_0}{\| P_0 \|_2} = \frac{1}{\sqrt{2}}
\]

\[
\phi_1 = \frac{P_1}{\| P_1 \|_2} = \frac{3}{\sqrt{2x}}
\]

\[
\phi_2 = \frac{P_2}{\| P_2 \|_2} = \frac{1}{2} \sqrt{\frac{5}{2}} (3x^2 - 1).
\]
(b) (15 pts) Determine \( a_0 \) and \( a_1 \) that satisfy

\[
\min_{a_0, a_1 \in \mathbb{R}} \int_{-1}^{1} |e^{-x} - (a_0 + a_1 x)|^2 \, dx.
\]

In other words, find the best line (i.e., the best linear approximation) to \( e^{-x} \) on \([-1,1]\) in the sense of the least squares.

**Answer:** Because \( \{\phi_n\}_{n=0}^{\infty} \) form an orthonormal basis for \( L^2[-1,1] \), the only thing we need is to expand \( e^x \) with respect to \( \phi_n \)'s we obtained in Part (a).

\[
\langle e^{-x}, \phi_0 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-x} \, dx = \frac{-e^{-1} + e^1}{\sqrt{2}} = \frac{e - e^{-1}}{\sqrt{2}}
\]

\[
\langle e^{-x}, \phi_1 \rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} xe^{-x} \, dx = \sqrt{\frac{3}{2}} \left\{ \left[-xe^{-x}\right]_{-1}^{1} + \int_{-1}^{1} e^{-x} \, dx \right\} \quad \text{Integration by Parts}
\]

\[
= \sqrt{\frac{3}{2}} \left\{ -e^{-1} - e^1 + \left(-e^{-1} + e^1\right) \right\}
\]

\[
= -\sqrt{6} e^{-1}
\]

Hence, the least squares line approximation to \( e^{-x} \) over \([-1,1]\) is:

\[
\frac{e - e^{-1}}{\sqrt{2}} \phi_0(x) - \sqrt{6} e^{-1} \phi_1(x) = \frac{e - e^{-1}}{\sqrt{2}} \frac{1}{\sqrt{2}} - \sqrt{6} e^{-1} \sqrt{\frac{3}{2}} x
\]

\[
= \frac{e - e^{-1}}{2} - 3e^{-1} x.
\]
Problem 4 (25 pts) Consider the following eigenvalue problem:

\[ u'' + \lambda u = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad \text{on } [0,1]. \]

Find the eigenvalues and normalized eigenfunctions.

Answer: First of all, we use the method of characteristic equation, i.e., assuming \( u \) is of the form \( e^{rx} \), we derive the algebraic equation in terms of \( r \). Clearly, we get \( r^2 + \lambda = 0 \). Thus, \( r^2 = -\lambda \). We need to consider the sign of \( \lambda \).

Case I: \( \lambda < 0 \). Then, we have \( r = \pm \sqrt{-\lambda} \in \mathbb{R} \). Thus, a solution in this case is \( u(x) = Ae^{\sqrt{-\lambda} x} + Be^{-\sqrt{-\lambda} x} \) where \( A, B \) are some constants. Then, \( u'(x) = \sqrt{-\lambda} \left( Ae^{\sqrt{-\lambda} x} - Be^{-\sqrt{-\lambda} x} \right) \). Thus, \( u'(0) = 0 \) gives us \( A - B = 0 \) because \( \sqrt{-\lambda} \neq 0 \). Now, \( u(1) = 0 \) gives us \( Ae^{\sqrt{-\lambda} 1} + Be^{-\sqrt{-\lambda} 1} = 0 \). Clearly, the only possibility is \( A = B = 0 \). Thus, this is a trivial solution, and cannot be considered as an eigenfunction. So, \( \lambda \) cannot be negative.

Case II: \( \lambda = 0 \). Then, the original ODE reduces to \( u'' = 0 \). Integrating twice, we have \( u(x) = Ax + B \) where \( A, B \) some constants. Now using the boundary conditions \( u'(0) = u(1) = 0 \), we can easily show that \( A = B = 0 \). Thus, \( \lambda \) cannot be 0.

Case III: \( \lambda > 0 \). Then, we have \( r = \pm \sqrt{\lambda} i \), i.e., pure imaginary numbers. Thus a solution can be written as:

\[
\begin{align*}
    u(x) &= Ae^{i\sqrt{\lambda} x} + Be^{-i\sqrt{\lambda} x} \\
          &= C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x
\end{align*}
\]

Now, since

\[
    u'(x) = -C_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + C_2 \sqrt{\lambda} \cos \sqrt{\lambda} x,
\]

the boundary condition \( u'(0) = 0 \) immediately gives us \( C_2 = 0 \). On the other hand, \( u(1) = 0 \) gives us

\[
    0 = C_1 \cos \sqrt{\lambda}
\]

Since \( C_1 \) cannot be 0 and \( \lambda \neq 0 \) (otherwise, the solution becomes the trivial solution), we must have \( \cos \sqrt{\lambda} = 0 \), i.e., \( \sqrt{\lambda} = (n + \frac{1}{2}) \pi \) where \( n \in \mathbb{Z} \). Thus we have \( u(x) = C_1 \cos \left(n + \frac{1}{2}\right) \pi x \). Now, the case for \( n < 0 \) can be absorbed to \( n > 0 \) case because of the evenness of \( \cos \) function. So, we have the eigenfunctions \( u(x) = C_1 \cos \left(n + \frac{1}{2}\right) \pi x, \ n = 0,1,2,... \). So the eigenvalues of this problem is:

\[
    \lambda = \left(n + \frac{1}{2}\right)^2 \pi^2, \ n = 0,1,\ldots
\]

In order to have a unit norm, we first compute

\[
    \int_0^1 \left( \cos \left(n + \frac{1}{2}\right) \pi x \right)^2 \, dx = \int_0^1 \frac{1 + \cos(2n + 1)\pi x}{2} \, dx = \frac{1}{2} \left[ x + \frac{\sin(2n + 1)\pi x}{(2n + 1)\pi} \right]_0^1 = \frac{1}{2}.
\]

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Hence, $\sqrt{2} \cos \left( n + \frac{1}{2} \right) \pi x$ has the unit norm. Thus, the normalized eigenfunctions are:

$$\sqrt{2} \cos \left( n + \frac{1}{2} \right) \pi x, \quad n = 0, 1, 2, \ldots.$$