MAT 129: Fourier Analysis
Supplementary Notes II by Naoki Saito

The Fourier Inversion Theorem

- The Fourier transform $\mathcal{F}$ was defined initially on $L^1(\mathbb{R})$, a space of integrable functions, and $\mathcal{F} : L^1(\mathbb{R}) \to BC(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
- However, $\widehat{f}$, the Fourier transform of $f \in L^1$, may not be in L^1.
  An example: $f(x) = \chi_{(-\frac{1}{2},\frac{1}{2})}(x) \Rightarrow \hat{f}(\xi) = \operatorname{sinc}(\xi) = \frac{\sin \pi\xi}{\pi\xi} \notin L^1$.
- **The Inverse Fourier Transform:** For $f \in L^1$, $\check{f}(x) = \displaystyle\int_{-\infty}^{\infty} f(\xi) \mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi$.
- **[The Fourier Inversion Theorem]** If both f and $\hat{f}$ are in L^1, then $(\hat{f})\check{} = (\check{f})\hat{} = f$ almost everywhere.
- There are many functions in L^1 whose Fourier transforms are also in L^1; one needs only a little *smoothness* of f for necessary *decay* of $\hat{f}$ as $|\xi| \to \infty$.
  An example: If $f \in C^2(\mathbb{R})$, f' and f'' are both in L^1, then $\mathcal{F}\{f''\}(\xi) = -(2\pi\xi)^2 \hat{f}(\xi) \in BC(\mathbb{R})$. This boundedness implies that $|\hat{f}(\xi)| \leq C/(1+\xi^2)$. This, in turn, implies that $\hat{f} \in L^1$.

The Fourier Transforms on L^2

- The previous remark leads to the L^2 theory of the Fourier transforms. In general, simply assuming $f \in L^2$ is not enough; $\displaystyle\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi \mathrm{i}\xi x}\,\mathrm{d}x$ may not converge.
  An example: $f(x) = \operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x} \in L^2$, but not in L^1.
- We will overcome this problem as follows. Define a subspace of L^1, $\mathcal{X} \triangleq \{f \in L^1 \mid \hat{f} \in L^1\}$. We first note that for such functions, we can have the Parseval equality: $\langle f, g\rangle = \left\langle \hat{f}, \hat{g} \right\rangle$ as well as the Plancherel equality. Also, for any $f \in \mathcal{X}$, $f, \hat{f} \in BC(\mathbb{R})$ as the remark after the Fourier inversion theorem. This implies that both f and $\hat{f}$ are also in L^2; i.e., $\mathcal{X} \subset L^2$ (because $f \in L^1 \cap BC$ implies $f \in L^2$ thanks to the theorem: $L^p \cap L^r \subset L^q$ for $0 < p < q < r \leq \infty$, which in turn can be proved by Hölder's inequality). Now, the point is that $\mathcal{X}$ is also *dense* in L^2.
- We can proceed as follows: for any $f \in L^2$, we can find a sequence $\{f_n\} \subset \mathcal{X}$ such that $\|f_n - f\|_2 \to 0$ as $n \to \infty$. $\{f_n\} \subset \mathcal{X}$ means that $\{\hat{f}_n\} \subset \mathcal{X}$. Now using the Plancherel equality to this sequence, we can see $\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2 \to 0$ as $m, n \to \infty$. In other words, $\{\hat{f}_n\}$ is a *Cauchy sequence* in L^2. Since L^2 is *complete*, there exists the limit of $\hat{f}_n$ in L^2, and we *define* this limit as $\hat{f}$, the Fourier transform of $f \in L^2$.
- **[The Plancherel Theorem]** For any $f, g \in L^2$, $\langle f, g\rangle = \left\langle \hat{f}, \hat{g} \right\rangle$ and $\|f\|_2 = \|\hat{f}\|_2$.
- Finally, we can use all these facts for computing the Fourier transform of L^2 functions as follows: Suppose we set $\phi(x) = \hat{f}(x)$ where $\hat{f} \in L^2$. Then, $\hat{\phi}(\xi) = f(-\xi)$. An example: $\phi(x) = \operatorname{sinc}(x) \in L^2$. Then, $\hat{\phi}(\xi) = \chi_{(-\frac{1}{2},\frac{1}{2})}(\xi)$.