Problem 1 (30 pts)

(a) (10 pts) Compute the Fourier series of $f(\theta) = \theta$ on $[-\pi, \pi]$, which is 2π periodic.

Answer: Since this is an odd function, we only need the coefficients with the sine terms, not the cosine terms. Hence, we need to compute, for $n \in \mathbb{N}$,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \theta \sin n\theta \, \mathrm{d}\theta \\ &= \frac{2}{\pi}\int_{0}^{\pi} \theta \sin n\theta \, \mathrm{d}\theta \\ &= \frac{2}{\pi}\left\{\left[\theta \cdot \frac{-\cos n\theta}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos n\theta \, \mathrm{d}\theta\right\} \quad \text{via Integration by Parts} \\ &= \frac{2}{\pi}\left\{\pi \frac{(-1)^{n+1}}{n} + \frac{1}{n}\left[\frac{\sin n\theta}{n}\right]_0^{\pi}\right\} \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Hence, we have

$$\boxed{f(\theta) = \theta \sim \sum_{n=1}^{\infty} b_n \sin n\theta = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta.}$$

(b) (10 pts) Compute the Fourier *cosine* series of $f(\theta) = \theta$ on $[0, \pi]$. Compare the decay rate of the coefficients with that of Part (a). Which coefficients decay faster?

Answer: Using the formula for the Fourier cosine coefficients over $[0, \pi]$, we have, for $n = 1, \ldots,$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{0}^{\pi} \theta \cos n\theta \, \mathrm{d}\theta \\ &= \frac{2}{\pi}\left\{\left[\theta \cdot \frac{\sin n\theta}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin n\theta \, \mathrm{d}\theta\right\} \quad \text{via Integration by Parts} \\ &= \frac{2}{\pi}\left\{0 - \frac{1}{n}\left[\frac{-\cos n\theta}{n}\right]_0^{\pi}\right\} \\ &= \frac{2}{\pi}\frac{(-1)^n - 1}{n^2}. \end{aligned}$$

Now, for $n = 0$, we have:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \theta \, \mathrm{d}\theta = \frac{2}{\pi}\left(\frac{\pi^2}{2} - 0\right) = \pi.$$

Hence, we have

$$\boxed{f(\theta) = \theta \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\theta = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2}.}$$

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As for the decay rates, the solution of Part a gives us $b_n = O\left(\frac{1}{n}\right)$ whereas that of Part b is $a_n = O\left(\frac{1}{n^2}\right)$, which clearly decays faster than the former. This of course comes from the functions to be expanded into these Fourier series. In Part a, $f(\theta) = \theta$ on $[-\pi, \pi]$ and extended periodically over the entire $\mathbb{R}$. This periodic function is *discontinuous* at $\theta = n\pi$, $\forall n \in \mathbb{Z}$. On the other hand, in Part b, $f(\theta) = \theta$ on $[0, \pi]$ is extended as an even function because of the Fourier cosine series expansion. Therefore, it is the same as the ordinary Fourier series expansion of $f(\theta) = |\theta|$ on $[-\pi, \pi]$, which is *continuous* over $\mathbb{R}$ even after periodic extension.

(c) (10 pts) Prove

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

using the result of Part (b).

Answer: We evaluate the Fourier cosine series of Part b at $\theta = 0$. Because the function $f(\theta) = \theta$ on $[-\pi, \pi]$ from the viewpoint of the Fourier cosine series expansion is the same as $f(\theta) = |\theta|$ on $[-\pi, \pi]$ from the viewpoint of the ordinary Fourier series expansion as discussed in Part b, we can use the pointwise convergent theorem of the Fourier series, since $f(\theta) = |\theta|$ after periodization with period 2π is clearly *piecewise smooth*. Hence,

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)0)}{(2n-1)^2} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

This leads to

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Now,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} + \frac{1}{(2n)^2} \right) && \text{Splitting even and odd terms} \\
&= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} && \text{since both series are convergent} \\
&= \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}$$

Let $S \triangleq \sum_{n=1}^{\infty} \frac{1}{n^2}$, then the above equation says:

$$S = \frac{\pi^2}{8} + \frac{S}{4} \Longleftrightarrow S = \frac{\pi^2}{6}. \quad \square$$

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Problem 2 (20 pts) Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be an orthonormal *set* in $L^2[a,b]$.

(a) (10 pts) For any function $f \in L^2[a,b]$, state *Bessel's inequality* for this function.

Answer: Bessel's inequality states:

$$\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \le \|f\|_2^2,$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$ are the inner product and the L^2-norm defined on $L^2[a,b]$.

(b) (10 pts) Under what condition Bessel's inequality becomes *Parseval's equality*?

Answer: Bessel's inequality always holds for any orthonormal *set*. If this set becomes an orthonormal *basis*, then the equality holds and becomes Parseval's equality. This can also be checked whether $\langle f, \phi_n \rangle = 0$ for all $n \in \mathbb{N}$ implies $f \equiv 0$ almost everywhere or not.

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Problem 3 (25 pts) The nth Legendre polynomial is defined as

$$P_n(x) \triangleq \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n, \quad n = 0, 1, \dots$$

The set $\{P_n\}_{n=0}^{\infty}$ form an *orthogonal* basis for $L^2[-1,1]$. Thus any $f \in L^2[-1,1]$ can be written as:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x).$$

From the above definition, we also know that $P_n(x)$ is a polynomial of degree n. Thus, a monomial x^M can be always written as $x^M = \sum_{n=0}^{M} a_n P_n(x)$, which is called the *Legendre expansion* of x^M.

(a) (10 pts) Obtain the Legendre expansion of x^2.

Answer: First of all, from the formula, we get: $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. Let us now write

$$x^2 = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x).$$

Now, take an inner product of this equation with $P_k(x)$, $k = 0, 1, 2$ gives us

$$\langle x^2, P_k \rangle = a_k \|P_k\|_2^2$$

thanks to the orthogonality. We know that $\|P_k\|_2^2 = 2/(2k+1)$ for all k (which can be derived too). Thus,

$$a_0 = \frac{\langle x^2, P_0 \rangle}{\|P_0\|_2^2} = \frac{1}{2}\int_{-1}^{1} x^2 \cdot 1 \,\mathrm{d}x = \frac{1}{3}.$$

$$a_1 = \frac{\langle x^2, P_1 \rangle}{\|P_1\|_2^2} = \frac{3}{2}\int_{-1}^{1} x^2 \cdot x \,\mathrm{d}x = 0.$$

$$a_2 = \frac{\langle x^2, P_0 \rangle}{\|P_2\|_2^2} = \frac{5}{2}\int_{-1}^{1} x^2 \cdot \frac{1}{2}(3x^2-1) \,\mathrm{d}x = \frac{5}{2}\int_{0}^{1} (3x^4 - x^2) \,\mathrm{d}x = \frac{5}{2}\left(\frac{3}{5} - \frac{1}{3}\right) = \frac{2}{3}.$$

Hence we have:

$$\boxed{x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x).}$$

(b) (15 pts) Let $\mathcal{P}_1$ be a set of all possible polynomial of degree 1, i.e., a set of all possible straight lines in $\mathbb{R}^2$. What is the *best linear L^2-approximation* to x^2 in $\mathcal{P}_1$ over the interval $[-1,1]$? In other words, what is the least squares line to approximate x^2 over $[-1,1]$?

Answer: We know that the Nth partial sum of an orthonormal expansion of a function in $L^2[-1,1]$ is the best linear approximation in the sense of the least squares among the subspace spanned by those first N basis functions. Therefore, in this case, $N = 2$, i.e., using P_0 and P_1, we have the expansion,

$$\frac{\langle x^2, P_0 \rangle}{\|P_0\|_2^2} P_0 + \frac{\langle x^2, P_1 \rangle}{\|P_1\|_2^2} P_1 = \boxed{\frac{1}{3}.}$$

Hence, in this case, simply the constant $y = \frac{1}{3}$ (i.e., a horizontal line) is better than any other line with nonzero slope.

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Problem 4 (25 pts) Find the eigenvalues and normalized eigenfunctions for the problem

$$u'' + \lambda u = 0, \quad u'(0) = u'(1) = 0, \quad \text{on } [0,1].$$

Answer: First of all, we use the method of characteristic equation, i.e., assuming u is of the form e^{rx}, we derive the algebraic equation in terms of r. Clearly, we get $r^2 + \lambda = 0$. Thus, $r^2 = -\lambda$. We need to consider the sign of λ.

Case I: $\lambda < 0$. Then, we have $r = \pm\sqrt{-\lambda} \in \mathbb{R}$. Thus, a solution in this case is $u(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ where A, B are some constants. Then, $u'(x) = \sqrt{-\lambda}\left(Ae^{\sqrt{-\lambda}x} - Be^{-\sqrt{-\lambda}x}\right)$. Thus, $u'(0) = 0$ gives us $A = B$ since $\sqrt{-\lambda} \neq 0$. Now, $u'(1) = 0$ gives us $A\left(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}\right) = 0$. Clearly, the only possibility is $A = B = 0$. Thus, this is a trivial solution, and cannot be considered as an eigenfunction. So, λ cannot be negative.

Case II: $\lambda = 0$. Then, the original ODE reduces to $u'' = 0$. Integrating twice, we have $u(x) = Ax + B$ where A, B some constants. Now using the boundary conditions $u'(0) = u'(1) = 0$, we can easily show that $A = 0$. Thus, $\lambda = 0$ and $u(x) = B \neq 0$ form a pair of an eigenvalue and the corresponding eigenfunction. Since $\|B\| = |B|$, the normalized eigenfunction is $u(x) = 1$.

Case III: $\lambda > 0$. Then, we have $r = \pm\sqrt{\lambda}i$, i.e., pure imaginary numbers. Thus a solution can be written as:

$$\begin{aligned} u(x) &= Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} \\ &= C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x \end{aligned}$$

Now, since

$$u'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x,$$

the boundary condition $u'(0) = 0$ immediately gives us $C_2 = 0$ since $\lambda \neq 0$. On the other hand, $u'(1) = 0$ gives us

$$0 = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}$$

Since $C_1 \neq 0$ and $\lambda \neq 0$ (otherwise, the solution becomes the trivial solution), we must have $\sin\sqrt{\lambda} = 0$, i.e., $\sqrt{\lambda} = n\pi$ where $n \in \mathbb{Z} \setminus \{0\}$. Thus we have $u(x) = C_1 \cos n\pi x$. Now, the case for $n < 0$ can be absorbed to $n > 0$ case by changing the sign of C_1. So, we have the eigenfunctions $u(x) = C_1 \cos n\pi x$, $n \in \mathbb{N}$. In order to have a unit norm, we compute for $n \geq 1$:

$$\int_0^1 (\cos n\pi x)^2 \, dx = \int_0^1 \frac{1 + \cos 2n\pi x}{2} \, dx = \frac{1}{2}\left[x + \frac{\sin 2n\pi x}{2n\pi}\right]_0^1 = \frac{1}{2}.$$

Hence, the $\sqrt{2}\cos n\pi x$ has the unit norm.

Thus, summarizing the results of Cases II and III, the eigenvalues and the normalized eigenfunctions for this problem are:

$$\boxed{\lambda_n = (n\pi)^2, \quad n = 0, 1, \ldots} \qquad \boxed{u_0(x) = 1, u_n(x) = \sqrt{2}\cos n\pi x, \quad n = 1, 2, \ldots}$$

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