Problem 1 (30 pts)

(a) (10 pts) Compute the Fourier series of $f(\theta) = \theta^2$ on $(-\pi, \pi)$, which is 2π periodic.

Answer: Since this function is an even function over $(-\pi, \pi)$, the Fourier series becomes a Fourier cosine series. Thus, we only need the Fourier cosine coefficients a_n, $n = 0, 1, 2, \ldots$. For a_n, $n \geq 1$, we have:

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi \theta^2 \cos n\theta \, \mathrm{d}\theta \\
&= \frac{2}{\pi}\left\{\left[\frac{\theta^2 \sin n\theta}{n}\right]_0^\pi - \frac{1}{n}\int_0^\pi 2\theta \sin n\theta \, \mathrm{d}\theta\right\} \quad \text{(Integration by Parts)} \\
&= -\frac{4}{n\pi}\int_0^\pi \theta \sin n\theta \, \mathrm{d}\theta \\
&= -\frac{4}{n\pi}\left\{\left[-\frac{\theta \cos n\theta}{n}\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos n\theta \, \mathrm{d}\theta\right\} \\
&= -\frac{4}{n\pi} \cdot \frac{\pi(-1)^{n+1}}{n} \\
&= 4\frac{(-1)^n}{n^2}.
\end{aligned}$$

Now, a_0 can be computed as:

$$a_0 = \frac{2}{\pi}\int_0^\pi \theta^2 \, \mathrm{d}\theta = \frac{2\pi^2}{3}.$$

Hence we have

$$\boxed{\theta^2 \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx = \frac{\pi^2}{3} + 4\sum_{n=1}^\infty \frac{(-1)^n}{n^2}\cos n\theta}.$$

(b) (10 pts) Prove

$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

using the result of Part (a).

Answer: Evaluate the Fourier series of $f(\theta)$ of Part (a) at $\theta = \pi$. (Another easy way is to evaluate it at $\theta = 0$, but I omit the proof for $\theta = 0$ case here). Since $\theta = \pi$ is a point of continuity, we have

$$\begin{aligned}
f(\pi) &= \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^\infty \frac{(-1)^n}{n^2}\cos n\pi \\
&= \frac{\pi^2}{3} + 4\sum_{n=1}^\infty \frac{(-1)^{2n}}{n^2} \\
&= \frac{\pi^2}{3} + 4\sum_{n=1}^\infty \frac{1}{n^2}.
\end{aligned}$$

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Hence, it is easy to derive

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) = \frac{\pi^2}{6}. \quad \square$$

(c) (10 pts) Using the result of Part (a), compute the Fourier series of $g(\theta) = \theta$ on $(-\pi, \pi)$, which is also 2π periodic.

Hint: Use the derivative formula.

Answer: Because the 2π periodic function θ^2 is *continuous and piecewise smooth* over $\mathbb{R}$, we can differentiate the result of Part (a) as:

$$2\theta \sim 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(-\sin n\theta),$$

from which we can easily derive:

$$\boxed{\theta \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta.}$$

Problem 2 (20 pts) Consider a function space, $L^2[-1,1]$.

(a) (4 pts) Assuming each function in $L^2[-1,1]$ is complex-valued, i.e., $f:[-1,1]\to\mathbb{C}$, state the standard definition of the L^2-norm and the inner product of this space.

Answer: Clearly, the standard L^2-norm for a function $f\in L^2[-1,1]$ is:

$$\|f\|_2 \triangleq \left(\int_{-1}^{1} |f(x)|^2\,\mathrm{d}x\right)^{1/2},$$

and the inner product of $f,g\in L^2[-1,1]$ is defined as:

$$\langle f,g\rangle \triangleq \int_{-1}^{1} f(x)\overline{g(x)}\,\mathrm{d}x.$$

(b) (4 pts) State the Cauchy-Schwarz inequality for this space.

Answer: The Cauchy-Schwarz inequality for $L^2[-1,1]$ is, for any $f,g\in L^2[-1,1]$,

$$\left|\langle f,g\rangle\right| \le \|f\|_2\|g\|_2,$$

where the equality holds if and only if f is proportional to g almost everywhere.

(c) (6 pts) The space $L^2[-1,1]$ is known to be *complete* with respect to the L^2-norm. State the definition of the completeness of this space.

Answer: Let $\{f_n\}\subset L^2[-1,1]$ be a Cauchy sequence in $L^2[-1,1]$, i.e., $\|f_m - f_n\|_2\to 0$ as $m,n\to\infty$. The completeness means that every Cauchy sequence in $L^2[-1,1]$ is a convergent sequence, i.e., there exists $f\in L^2[-1,1]$ such that $\|f_n - f\|_2\to 0$ as $n\to\infty$.

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(d) (6 pts) Show an example of an orthonormal *set* in $L^2[-1,1]$, which is not *complete* (i.e., not an orthonormal *basis* for $L^2[-1,1]$).

Answer: Consider the following set of functions:

$$\phi_n(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0, \\ \sqrt{2}\sin n\pi x & \text{if } 0 \le x \le 1. \end{cases} \qquad n = 1, 2, \ldots$$

Then, this set of function is an orthonormal set because:

$$\begin{aligned} \langle \phi_m, \phi_n \rangle &= \int_{-1}^{1} \phi_m(x)\phi_n(x)\,\mathrm{d}x \\ &= \int_0^1 2\sin m\pi x \sin n\pi x\,\mathrm{d}x \\ &= \begin{cases} \int_0^1 (\cos(m-n)\pi x - \cos(m+n)\pi x)\,\mathrm{d}x & \text{if } m \ne n; \\ \int_0^1 (1 - \cos 2m\pi x)\,\mathrm{d}x & \text{if } m = n \end{cases} \\ &= \begin{cases} \left[\frac{\sin(m-n)\pi x}{(m-n)\pi} - \frac{\sin(m+n)\pi x}{(m+n)\pi}\right]_0^1 & \text{if } m \ne n; \\ \left[x - \frac{\sin 2m\pi x}{2m\pi}\right]_0^1 & \text{if } m = n \end{cases} \\ &= \delta_{mn}. \end{aligned}$$

However, this set clearly cannot represent all the functions in $L^2[-1,1]$ since $\phi_n(x) = 0$ for $n = 0, 1, \ldots$, i.e., $\{\phi_n\}_1^\infty$ cannot satisfy *Parseval's equality* for functions in $L^2[-1,1]$. Therefore, $\{\phi_n\}_1^\infty$ is not a complete orthonormal set in $L^2[-1,1]$.

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Problem 3 (25 pts) The first three Legendre polynomials are:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1).$$

(a) (10 pts) Show that they are mutually orthogonal. Moreover, compute their ortho*normal* version, $\phi_0(x), \phi_1(x), \phi_2(x)$.

Answer: They are mutually orthogonal because:

$$\begin{aligned}
\langle P_0, P_1 \rangle &= \int_{-1}^{1} 1 \cdot x \, \mathrm{d}x = 0 \quad \text{because } x \text{ is an odd function.} \\
\langle P_0, P_2 \rangle &= \int_{-1}^{1} 1 \cdot \frac{1}{2}(3x^2 - 1) \, \mathrm{d}x = \frac{1}{2} \left[x^3 - x \right]_{-1}^{1} = 0 \\
\langle P_1, P_2 \rangle &= \int_{-1}^{1} x \cdot \frac{1}{2}(3x^2 - 1) \, \mathrm{d}x = 0 \quad \text{because } x^3 \text{ and } x \text{ are odd functions.}
\end{aligned}$$

Now, let's compute the L^2-norm of them.

$$\begin{aligned}
\|P_0\|_2 &= \sqrt{\int_{-1}^{1} 1^2 \, \mathrm{d}x} = \sqrt{2} \\
\|P_1\|_2 &= \sqrt{\int_{-1}^{1} x^2 \, \mathrm{d}x} = \sqrt{\frac{2}{3}} \\
\|P_2\|_2 &= \sqrt{\int_{-1}^{1} \frac{1}{4}(3x^2 - 1)^2 \, \mathrm{d}x} = \sqrt{\frac{1}{2} \int_{0}^{1} (9x^4 - 6x^2 + 1) \, \mathrm{d}x} = \sqrt{\frac{1}{2}\left(\frac{9}{5} - 2 + 1\right)} = \sqrt{\frac{2}{5}}.
\end{aligned}$$

Note that one can also use the formula $\|P_n\|_2 = \sqrt{\frac{2}{2n+1}}$, $n = 0, 1, 2, \ldots$. Therefore, we have:

$$\begin{aligned}
\phi_0 &= \frac{P_0}{\|P_0\|_2} = \boxed{\frac{1}{\sqrt{2}}} \\
\phi_1 &= \frac{P_1}{\|P_1\|_2} = \boxed{\sqrt{\frac{3}{2}} x} \\
\phi_2 &= \frac{P_2}{\|P_2\|_2} = \boxed{\frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1)}.
\end{aligned}$$

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(b) (15 pts) Determine a_0 and a_1 that satisfy

$$\min_{a_0, a_1 \in \mathbb{R}} \int_{-1}^{1} |\mathrm{e}^{-x} - (a_0 + a_1 x)|^2 \, \mathrm{d}x.$$

In other words, find the best line (i.e., the best linear approximation) to e^{-x} on $[-1,1]$ in the sense of the least squares.

Answer: Because $\{\phi_n\}_0^\infty$ form an orthonormal basis for $L^2[-1,1]$, the only thing we need is to expand e^x with respect to ϕ_n's we obtained in Part (a).

$$\begin{aligned}
\left\langle \mathrm{e}^{-x}, \phi_0 \right\rangle &= \frac{1}{\sqrt{2}} \int_{-1}^{1} \mathrm{e}^{-x} \, \mathrm{d}x = \frac{-\mathrm{e}^{-1} + \mathrm{e}^{1}}{\sqrt{2}} = \frac{\mathrm{e} - \mathrm{e}^{-1}}{\sqrt{2}} \\
\left\langle \mathrm{e}^{-x}, \phi_1 \right\rangle &= \sqrt{\frac{3}{2}} \int_{-1}^{1} x \mathrm{e}^{-x} \, \mathrm{d}x = \sqrt{\frac{3}{2}} \left\{ \left[-x \mathrm{e}^{-x} \right]_{-1}^{1} + \int_{-1}^{1} \mathrm{e}^{-x} \, \mathrm{d}x \right\} \quad \text{Integration by Parts} \\
&= \sqrt{\frac{3}{2}} \left\{ -\mathrm{e}^{-1} - \mathrm{e}^{1} + (-\mathrm{e}^{-1} + \mathrm{e}^{1}) \right\} \\
&= -\sqrt{6} \mathrm{e}^{-1}
\end{aligned}$$

Hence, the least squares line approximation to e^{-x} over $[-1,1]$ is:

$$\begin{aligned}
\frac{\mathrm{e} - \mathrm{e}^{-1}}{\sqrt{2}} \phi_0(x) - \sqrt{6} \mathrm{e}^{-1} \phi_1(x) &= \frac{\mathrm{e} - \mathrm{e}^{-1}}{\sqrt{2}} \frac{1}{\sqrt{2}} - \sqrt{6} \mathrm{e}^{-1} \sqrt{\frac{3}{2}} x \\
&= \boxed{\frac{\mathrm{e} - \mathrm{e}^{-1}}{2} - 3 \mathrm{e}^{-1} x}.
\end{aligned}$$

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Problem 4 (25 pts) Consider the following eigenvalue problem:

$$u'' + \lambda u = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad \text{on } [0,1].$$

Find the eigenvalues and normalized eigenfunctions.

Answer: First of all, we use the method of characteristic equation, i.e., assuming u is of the form e^{rx}, we derive the algebraic equation in terms of r. Clearly, we get $r^2 + \lambda = 0$. Thus, $r^2 = -\lambda$. We need to consider the sign of λ.

Case I: $\lambda < 0$. Then, we have $r = \pm\sqrt{-\lambda} \in \mathbb{R}$. Thus, a solution in this case is $u(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ where A, B are some constants. Then, $u'(x) = \sqrt{-\lambda}\left(Ae^{\sqrt{-\lambda}x} - Be^{-\sqrt{-\lambda}x}\right)$. Thus, $u'(0) = 0$ gives us $A - B = 0$ because $\sqrt{-\lambda} \neq 0$. Now, $u(1) = 0$ gives us $Ae^{\sqrt{-\lambda}} + Be^{-\sqrt{-\lambda}} = 0$. Clearly, the only possibility is $A = B = 0$. Thus, this is a trivial solution, and cannot be considered as an eigenfunction. So, λ cannot be negative.

Case II: $\lambda = 0$. Then, the original ODE reduces to $u'' = 0$. Integrating twice, we have $u(x) = Ax + B$ where A, B some constants. Now using the boundary conditions $u'(0) = u(1) = 0$, we can easily show that $A = B = 0$. Thus, λ cannot be 0.

Case III: $\lambda > 0$. Then, we have $r = \pm\sqrt{\lambda}i$, i.e., pure imaginary numbers. Thus a solution can be written as:

$$\begin{aligned} u(x) &= Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} \\ &= C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x \end{aligned}$$

Now, since

$$u'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x,$$

the boundary condition $u'(0) = 0$ immediately gives us $C_2 = 0$. On the other hand, $u(1) = 0$ gives us

$$0 = C_1 \cos\sqrt{\lambda}$$

Since C_1 cannot be 0 and $\lambda \neq 0$ (otherwise, the solution becomes the trivial solution), we must have $\cos\sqrt{\lambda} = 0$, i.e., $\sqrt{\lambda} = \left(n + \frac{1}{2}\right)\pi$ where $n \in \mathbb{Z}$. Thus we have $u(x) = C_1 \cos\left(n + \frac{1}{2}\right)\pi x$. Now, the case for $n < 0$ can be absorbed to $n > 0$ case because of the evenness of cos function. So, we have the eigenfunctions $u(x) = C_1 \cos\left(n + \frac{1}{2}\right)\pi x$, $n = 0, 1, 2, \ldots$. So the eigenvalues of this problem is:

$$\boxed{\lambda = \left(n + \frac{1}{2}\right)^2 \pi^2, \quad n = 0, 1, \ldots}$$

In order to have a unit norm, we first compute

$$\int_0^1 \left(\cos\left(n + \frac{1}{2}\right)\pi x\right)^2 dx = \int_0^1 \frac{1 + \cos(2n+1)\pi x}{2} dx = \frac{1}{2}\left[x + \frac{\sin(2n+1)\pi x}{(2n+1)\pi}\right]_0^1 = \frac{1}{2}.$$

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Hence, $\sqrt{2}\cos\left(n+\frac{1}{2}\right)\pi x$ has the unit norm. Thus, the normalized eigenfunctions are:

$$\boxed{\sqrt{2}\cos\left(n+\frac{1}{2}\right)\pi x, \quad n=0,1,2,\ldots.}$$

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