MATH 132A: Intro. to Stochastic Processes
Final Exam Solutions

Name:_________________________
Score of this problem:___________

Total Score:__________________

Problem 1 (25 pts) Consider a Poisson process with rate λ. Let T_1 be the arrival time of the first event, T_n be the interarrival time between the $(n-1)$st and the nth events.

(a) (10 pts) Show that $P\{T_1 > t\} = \mathrm{e}^{-\lambda t}$. [Hint: Recall the definition of a Poisson process.]

Ans: Clearly, $P\{T_1 > t\} = P\{N(t) = 0\}$. From the definition of a Poisson process,

$$P\{N(t+s) - N(s) = n\} = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!},$$

for all $t \geq 0$, all $s \geq 0$, and all $n = 0, 1, \ldots$. So, taking $s = 0$, $n = 0$, we get $P\{N(t) = 0\} = \mathrm{e}^{-\lambda t}$.

(b) (10 pts) Show that $P\{T_2 > t\} = \mathrm{e}^{-\lambda t}$.

Ans: Conditioning on T_1, we have

$$P\{T_2 > t\} = E[P\{T_2 > t \,|\, T_1\}].$$

However,

$$\begin{aligned} P\{T_2 > t \,|\, T_1 = s\} &= P\{N(t+s) - N(s) = 0 \,|\, T_1 = s\} \\ &= P\{N(t+s) - N(s) = 0\} \\ &= \mathrm{e}^{-\lambda t}. \end{aligned}$$

(c) (5 pts) Using (a) and (b), prove that $\{T_n; n = 1, 2, \ldots\}$ are independently and identically distributed exponential random variables with parameter λ.

Ans: We can repeat the above argument for $T_3, T_4, \ldots$, we can easily conclude that $P\{T_n > t\} = \mathrm{e}^{-\lambda t}$ for $n = 0, 1, 2, \ldots$. So, they are independent and identically distributed. Finally, if an random variable X obeys an exponential distribution with parameter λ, its cdf is $F(t) = P\{X \leq t\} = 1 - e^{-\lambda t}$. Thus, $P\{X > t\} = \mathrm{e}^{-\lambda t}$. This implies that T_n obeys exponential distribution with parameter λ for $n = 0, 1, 2, \ldots$.

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Problem 2 (20 pts) For a Poisson process, show that for $s < t$,

$$P\{N(s) = k \mid N(t) = n\} = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}, \quad k = 0, 1, \ldots, n.$$

[Hint: $P\{N(s) = k \mid N(t) = n\} = P\{N(s) = k, N(t) = n\}/P\{N(t) = n\}$. Then consider how many events arrive in the interval $(s, t]$.]

Ans: Following the hint, we have

$$\begin{aligned}
P\{N(s) = k \mid N(t) = n\} &= \frac{P\{N(s) = k, N(t) = n\}}{P\{N(t) = n\}} \\
&= \frac{P\{N(s) = k, N(t-s) = n-k\}}{P\{N(t) = n\}} \\
&= \frac{P\{N(s) = k\} \cdot P\{N(t-s) = n-k\}}{P\{N(t) = n\}} \\
&= \frac{\mathrm{e}^{-\lambda s}(\lambda s)^k/k! \cdot \mathrm{e}^{-\lambda(t-s)}(\lambda(t-s))^{n-k}/(n-k)!}{\mathrm{e}^{-\lambda t}(\lambda t)^n/n!} \\
&= \frac{n!}{k!(n-k)!} \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^{n-k} \\
&= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}.
\end{aligned}$$

Score of this problem:____________

Problem 3 (30 pts) Consider three independent random variables X_1, X_2, X_3, which obey exponential distribution with parameters $\lambda_1, \lambda_2, \lambda_3$, respectively.

(a) (8 pts) What is $P\{X_1 < X_2\}$?
[Hint: Condition on X_1.]
Ans: Following the hint, we have

$$\begin{aligned} P\{X_1 < X_2\} &= E[P\{X_1 < X_2 \mid X_1\}] \\ &= \int_0^\infty P\{X_1 < X_2 \mid X_1 = x\} \lambda_1 \mathrm{e}^{-\lambda_1 x} \mathrm{d}x \\ &= \int_0^\infty P\{X_2 > x\} \lambda_1 \mathrm{e}^{-\lambda_1 x} \mathrm{d}x \\ &= \int_0^\infty \mathrm{e}^{-\lambda_2 x} \lambda_1 \mathrm{e}^{-\lambda_1 x} \mathrm{d}x \\ &= \lambda_1 \int_0^\infty \mathrm{e}^{-(\lambda_1+\lambda_2)x} \mathrm{d}x \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2}. \end{aligned}$$

(b) (7 pts) Compute $P\{\min(X_2, X_3) > x\}$, and what is the probability distribution of the random variable $\min(X_2, X_3)$?
Ans:

$$\begin{aligned} P\{\min(X_2, X_3) > x\} &= P\{X_2 > x \text{ and } X_3 > x\} \\ &= P\{X_2 > x\} \cdot P\{X_3 > x\} \quad \text{from independence;} \\ &= \mathrm{e}^{-\lambda_2 x} \cdot \mathrm{e}^{-\lambda_3 x} \\ &= \mathrm{e}^{-(\lambda_2+\lambda_3)x}. \end{aligned}$$

This implies that $\min(X_2, X_3)$ obeys exponential distribution with parameter $\lambda_2 + \lambda_3$.

(c) (7 pts) What is $P\{X_1 = \min(X_1, X_2, X_3)\}$?
[Hint: Consider $P\{X_1 < \min(X_2, X_3)\}$, and use (a) and (b).]
Ans: Thinking about $\min(X_2, X_3)$ is a single exponential random variable with parameter $\lambda_2 + \lambda_3$, we have

$$\begin{aligned} P\{X_1 = \min(X_1, X_2, X_3)\} &= P\{X_1 < \min(X_2, X_3)\} \\ &= \frac{\lambda_1}{\lambda_1 + (\lambda_2+\lambda_3)} \quad \text{from (a);} \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}. \end{aligned}$$

(d) (8 pts) What is $P\{X_1 < X_2 < X_3\}$?

[Hint:

$$P\{X_1 < X_2 < X_3\} = P\{X_1 = \min(X_1, X_2, X_3)\} \cdot P\{X_2 < X_3 \mid X_1 = \min(X_1, X_2, X_3)\}.$$

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Ans: Following the hint, we have:

$$\begin{aligned} P\{X_1 < X_2 < X_3\} &= P\{X_1 = \min(X_1, X_2, X_3)\} \cdot P\{X_2 < X_3 \mid X_1 = \min(X_1, X_2, X_3)\} \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot P\{X_2 < X_3 \mid X_1 = \min(X_1, X_2, X_3)\} \quad \text{from (c);} \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot P\{X_2 < X_3\} \quad \text{from the memoryless property;} \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3} \quad \text{from (a).} \end{aligned}$$

Score of this problem:____________

Problem 4 (25 pts) Consider a homogeneous Markov chain $\{X_n;\ n = 0, 1, \ldots\}$ whose state is *binary*, i.e., $X_n = 0$ or 1 for all $n \geq 0$. Let us denote $\alpha = P_{01}$ and $\beta = P_{10}$.

(a) (5 pts) Write the transition probability matrix P.
Ans: This is easy.

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

(b) (20 pts) Compute the limiting probabilities, i.e.,

$$\Pi = \lim_{n\to\infty} P^n.$$

[Hint: You need to classify the situation depending on whether $\alpha = 0$, $0 < \alpha < 1$, or $\alpha = 1$, so as β.]

Ans:

- $\alpha = \beta = 0$: Then $P = I$, i.e., the identity matrix. So, clearly, $\Pi = I$.
- $\alpha = \beta = 1$: Then $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So, the chain oscillates between state 0 and state 1 indefinitely. Therefore Π does not exist.
- $0 < \alpha \leq 1$ and $\beta = 0$: Then as soon as the chain reaches state 1, it sticks to it. So, $\Pi = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.
- $\alpha = 0$ and $0 < \beta \leq 1$: This is the opposite of the above case. So, $\Pi = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.
- $0 < \alpha < 1$ and $0 < \beta < 1$: This case, we can compute the stationary distributions $\boldsymbol{\pi} = (\pi_0, \pi_1)$, which satisfies $\boldsymbol{\pi} = \boldsymbol{\pi} P$. From this we get $\boldsymbol{\pi} = (\beta/(\alpha+\beta), \alpha/(\alpha+\beta))$. Therefore, we have

$$\Pi = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}$$

Score of this problem:____________

Problem 5 (20 pts) Consider a branching process. Let X_n be the population of the nth generation, and let μ be the expected number of offspring produced by an individual in this population. Let us assume $X_0 = 1$.

(a) (10 pts) Compute $E[X_n]$.

[Hint: Represent $X_n = \sum_{i=1}^{X_{n-1}} Z_i$, where Z_i is the number of offspring of the ith individual of the $(n-1)$st generation. Clearly, $E[Z_i] = \mu$.]

Ans:

$$\begin{aligned}
E[X_n] &= E[E[X_n \mid X_{n-1}]] \\
&= E\left[E\left[\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1}\right]\right] \\
&= E[X_{n-1}E[Z_i]] \\
&= \mu E[X_{n-1}] \\
&= \mu^2 E[X_{n-2}] \\
&= \cdots \\
&= \mu^n E[X_0] \\
&= \mu^n.
\end{aligned}$$

(b) (10 pts) Show

$$E[X_m X_n] = \mu^{n-m} E[X_m^2] \quad \text{for } m \le n.$$

[Hint: First consider $E[X_n \mid X_m]$. Next consider $E[X_m X_n \mid X_m]$.]

Ans: Let's follow the hint.

$$E[X_n \mid X_m] = X_m \, E[X_{n-m}], \tag{1}$$

since it is the same as starting X_m individuals and thinking $(n-m)$th generation. Now,

$$\begin{aligned}
E[X_m X_n] &= E[E[X_m X_n | X_m]] \quad \text{conditioning on } X_m; \\
&= E[X_m \, E[X_n \mid X_m]] \\
&= E[X_m^2 \, E[X_{n-m}]] \quad \text{from (1);} \\
&= E[X_m^2] \, E[X_{n-m}] \quad \text{since } E[X_{n-m}] \text{ is just a constant;} \\
&= \mu^{n-m} E[X_m^2] \quad \text{from (a).}
\end{aligned}$$