MAT 132A Homework assignment #1

Exercise 3.1

$$\sum_x P_{X|Y}(x|y) = \frac{\sum_x P(x,y)}{P_Y(y)} = \frac{P_Y(y)}{P_Y(y)} = 1$$

Exercise 3.3

$$E[X|Y=1] = 2$$
$$E[X|Y=2] = \frac{5}{3}$$
$$E[X|Y=3] = \frac{12}{5}$$

Exercise 3.4

No.

Exercise 3.8

a)

$E[X] = E[X|\text{first roll is 6}]\frac{1}{6} + E[X|\text{fist roll is not 6}]\frac{5}{6} = \frac{1}{6} + (1 + E[X])\frac{5}{6}$

implying that $E[X] = 6$

b) $E[X|Y=1] = 1 + E[X] = 7$

c)

$$\begin{aligned} E[X|Y=5] = 1\left(\frac{1}{5}\right) &+ 2\left(\frac{4}{5}\right)\left(\frac{1}{5}\right) + 3\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right) \\ &+ 4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right) + 6\left(\frac{4}{5}\right)^4\left(\frac{1}{6}\right) \\ &+ 7\left(\frac{4}{5}\right)^4\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \ldots \end{aligned}$$

Exercise 3.9

$$\begin{aligned} E[X|Y=y] &= \sum_x xP\{X=x|Y=y\} \\ &= \sum_x xP\{X=x\} \quad \text{by independence} \\ &= E[X] \end{aligned}$$

Exercise 3.14

$$\begin{aligned} f_{X|X<\frac{1}{2}}(x) &= \frac{f(x)}{P\{X<1/2\}}, x<\frac{1}{2} \\ &= \frac{1}{1/2} = 2 \end{aligned}$$

Hence, $E[X|X<\frac{1}{2}] = \int_0^{1/2} 2x\,dx = \frac{1}{4}$

Exercise 3.16

a) Let us first rewrite $f(x,y)$ in a more convenient form by "completing the square" in the exponent:

$$\begin{aligned}
f(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\right. \\
&\quad \left.\times[(\frac{x-\mu_x}{\sigma_x})^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + (\frac{y-\mu_y}{\sigma_y})^2]\right\} \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\} \\
&\quad \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_x}{\sigma_x} - \rho\frac{y-\mu_y}{\sigma_y}\right)^2\right\}
\end{aligned}$$

Now if we let

$$a = \frac{1}{2(1-\rho^2)} \qquad \text{and} \qquad z = \frac{x-\mu_x}{\sigma_x} - \rho\frac{y-\mu_y}{\sigma_y} \qquad \text{so that} \qquad dx = \sigma_x\,dz$$

it follows that

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2} \int_{-\infty}^{\infty} e^{-az^2}\sigma_x\,dz \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2} \sigma_x\sqrt{\frac{\pi}{a}} \\
&= \frac{1}{2\pi\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2} \sqrt{2\pi(1-\rho^2)} \\
&= \frac{1}{\sqrt{2\pi}\sigma_y}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2} \\
&= N(\mu_y, \sigma_y^2)
\end{aligned}$$

as was to be shown. In order to find $f_X(x)$ we can again rearrange $f(x,y)$ in the following convenient way, just as we did above:

$$
\begin{aligned}
f(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\,\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\} \\
&\quad \times\exp\left\{-\frac{1}{2(1-\rho^2)}(\frac{y-\mu_y}{\sigma_y}-\rho\frac{x-\mu_x}{\sigma_x})^2\right\}
\end{aligned}
$$

Then, letting:

$$
a=\frac{1}{2(1-\rho^2)} \qquad \text{and} \qquad z=\frac{y-\mu_y}{\sigma_y}-\rho\frac{x-\mu_x}{\sigma_x} \qquad \text{so that} \qquad dy=\sigma_y\,dz
$$

we get:

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\,e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\int_{-\infty}^{\infty} e^{-az^2}\sigma_y\,dz \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\,e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\sigma_y\sqrt{\frac{\pi}{a}} \\
&= \frac{1}{2\pi\sigma_x\sqrt{1-\rho^2}}\,e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\sqrt{2\pi(1-\rho^2)} \\
&= \frac{1}{\sqrt{2\pi}\sigma_x}\,e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \\
&= N(\mu_x,\sigma_x^2)
\end{aligned}
$$

b) In part a) we rewrote $f(x,y)$ as:

$$
\begin{aligned}
f(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\,\exp\left\{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\} \\
&\quad \times\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_x}{\sigma_x}-\rho\frac{y-\mu_y}{\sigma_y}\right)^2\right\}
\end{aligned}
$$

and we found that:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

Therefore:

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_x}{\sigma_x} - \rho\frac{y-\mu_y}{\sigma_y}\right)^2\right\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\frac{x-(\mu_x+(\rho\sigma_x/\sigma_y)(y-\mu_y))}{\sigma_x\sqrt{1-\rho^2}}\right)^2\right\} \\
&= N(\mu_x + (\rho\sigma_x/\sigma_y)(y-\mu_y), \sigma_x^2(1-\rho^2))
\end{aligned}$$

Exercise 3.20

a)

$$\begin{aligned}
f(x|\text{disease}) &= \frac{P\{\text{disease}|x\}f(x)}{\int P\{\text{disease}|x\}f(x)\,dx} \\
&= \frac{P\{x\}f(x)}{\int P\{x\}f(x)\,dx}
\end{aligned}$$

b)

$$f(x|\text{no disease}) = \frac{[1-P(x)]f(x)}{\int[1-P(x)]f(x)\,dx}$$

c)

$$\frac{f(x|\text{disease})}{f(x|\text{no disease})} = C\frac{P(x)}{1-P(x)}$$

where C does not depend on x.

Exercise 3.21

a) $X = \sum_{i=1}^{N} T_i$

b) Clearly N is geometric with parameter 1/3; thus $E[N] = 3$.

c) Since T_N is the travel time corresponding to the choice leading to freedom, it follows that $T_N = 2$, and so $E[T_N] = 2$.

d) Given that $N = n$, the travel times $T_i, i = 1, \ldots, n-1$ are each equally likely to be either 3 or 5 (since we know that a door leading back to the mine is selected), whereas T_n is equal to 2 (since that choice led to safety). Hence,

$$\begin{aligned} E[\sum_{i=1}^{N} T_i|N=n] &= E[\sum_{i=1}^{n-1} T_i|N=n] + E[T_n|N=n] \\ &= 4(n-1)+2 \end{aligned}$$

e) Since part d) is equivalent to the equation

$$E[\sum_{i=1}^{N} T_i|N] = 4N - 2$$

we see from parts a) and b) that

$$E[X] = 4E[N] - 2 = 10$$