MAT 132A Homework assignment #2

Exercise 3.24

In all parts, let X denote the random variable whose expectation is desired, and start by conditioning on the result of the first flip. Also, h stands for heads and t for tails.

(a)

$$\begin{aligned} E[X] &= E[X|h]\,p + E[X|t]\,(1-p) \\ &= (1 + \frac{1}{1-p})\,p + (1 + \frac{1}{p})(1-p) \\ &= 1 + p/(1-p) + (1-p)/p \end{aligned}$$

(b)

$$\begin{aligned} E[X] &= (1 + E[\text{numebr of heads before first tail}])\,p + 1\,(1-p) \\ &= 1 + p/(1-p) + (1-p)/p \end{aligned}$$

(c) Interchanging p and $1-p$ in (b) gives result: $1+(1-p)/p-(1-p)$

d)

$$\begin{aligned} E[X] &= (1 + \text{ answer from (a)})\,p + (1 + 2/p)(1-p) \\ &= (2 + p/(1-p) + (1-p)/p)\,p + (1 + 2/p)(1-p) \end{aligned}$$

Exercise 3.30

$$E[N] = \sum_{j=1}^{m} E[N|X_0 = j]\,p(j) = \sum_{j=1}^{m} \frac{1}{p(j)} p(j) = m$$

Exercise 3.34 Let U denote the point of the break. Then,

$$E[L(x)] = \int_0^1 E[L(x)]|U = u]\, du = \int_0^x (1-u)\, du + \int_x^1 u\, du = x - x^2 + 1/2$$

Calculus shows that it is maximized when $x = 1/2$.

Exercise 3.35

(a) $E[X] = (2.6 + 3 + 3.4)/3 = 3$

(b) $E[X^2] = (2.6 + 2.6^2 + 3 + 9 + 3.4 + 3.4^2)/3 = 12.1067$, and $\mathrm{Var}(X) = 3.1067$

Exercise 3.43

By definition of $\mathrm{Var}(X|Y))$, we have that

$$\begin{aligned}\mathrm{Var}(X|Y) &= E[X^2 - 2\,X\,E[X|Y] + E^2[X|Y](Y) \\ &= E[X^2|Y] - 2\,E[X|Y]\,E[X|Y] + E^2[X|Y],\end{aligned}$$

where we have used the fact that $E[X|Y]$ and $E^2[X|Y]$ are functions of Y and thus, given Y, they may be treated as constants. Therefore,

$$\mathrm{Var}(X|Y) = E[X^2|Y] - E^2[X|Y]$$

and taking expectations yields

$$\begin{aligned}E[\mathrm{Var}(X|Y)] &= E[E[X^2|Y]] - E[E^2[X|Y]] \quad (*) \\ &= E[X^2] - E[E^2[X|Y]]\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{Var}(X|Y) &= E[(E[X|Y] - E[E[X|Y]])^2] \\
&= E[(E[X|Y] - E[X])^2] \\
&= E[E^2[X|Y] - 2\,E[X]\,E[X|Y]] + E^2[X] \\
&= E[E^2[X|Y]] - 2\,E[X]\,E[E[X|Y]]] + E^2[X] \\
&= E[E^2[X|Y]] - 2\,E^2[X] + E^2[X] \\
&= E[E^2[X|Y]] - E^2[x]
\end{aligned}$$

Hence, from equation (*) and the above equation, we arrive at:

$$E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]) = E[X^2] - E^2[X]$$

Exercise 3.49

Assume that independently of the past, each fish caught is tossed back with probabiity $2/3.e^(- 10)(10)^n/n!$

Mean and variance of the number of fish caught both equal 30.

Mean and variance of the number he takes home both equal 10.

Exercise 3.50

$$\begin{aligned}
P(N = n) &= \frac{1}{3}\Bigg[\binom{10}{n}(.3)^n(.7)^{10-n} + \binom{10}{n}(.5)^n(.5)^{10-n} \\
&\quad + \binom{10}{n}(.7)^n(.3)^{10-n}\Bigg]
\end{aligned}$$

N is not binomial.

$$E[N] = 3\left(\frac{1}{3}\right) + 5\left(\frac{1}{3}\right) + 7\left(\frac{1}{3}\right) = 5$$

Exercise 3.55

Let us define the following r.v.'s:

$$X_k = \begin{cases} 1 & \text{if head,} \\ 0 & \text{if tail,} \end{cases}$$

for $k = 1, \dots, n$, and

$$S_n = \sum_{k=1}^{n} X_k,$$

that is, the number of heads in n coin tosses. Let us also define

$$S_n^j = \sum_{\substack{k=1 \\ k \neq j}}^{n} X_k,$$

i.e., the number of heads in n coin tosses excluding the jth toss. Then,

$$\begin{aligned} P\{S_n = \text{even}\} &= P\{X_j = 1 \mid S_n^j = \text{odd}\} P\{S_n^j = \text{odd}\} \\ &\quad + P\{X_j = 0 \mid S_n^j = \text{even}\} P\{S_n^j = \text{even}\} \\ &= P\{X_j = 1\} P\{S_n^j = \text{odd}\} \\ &\quad + P\{X_j = 0\} P\{S_n^j = \text{even}\} \quad \text{from independence} \\ &= \frac{1}{2}\left(1 - P\{S_n^j = \text{even}\}\right) + \frac{1}{2} P\{S_n^j = \text{even}\} \\ &= \frac{1}{2}. \end{aligned}$$

Note that this is not true if $P\{X_j = 1\} = P\{X_j = 0\} = 1/2$ does not hold.